Polarized Electorate and Turnout*

Kemal Kıvanç Aköz† Alexey Zakharov‡

August 4, 2020

Abstract

We analyze a strategic voting model with an electorate divided between a single pro-incumbent and multiple opposition camps, and look at the effect of preference polarization among the pro-opposition voters. If there are multiple opposition options (parties or candidates), the turnout among the opposition will be smaller if the cohesion among the pro-opposition voters is higher, due to a free-rider effect; this is true for both proportional representation and a winner-take-all system. If there is a single pro-opposition option, the effect will be the opposite under proportional representation, and will depend on the relative sizes of pro-incumbent and pro-opposition electorates under a winner-tale-all system. Finally, we find that facing a single pro-opposition option might increase the likelihood of the pro-incumbent outcome for proportional representation and also for the winner-take-all system if the relative size of the pro-incumbent electorate is large enough.

JEL classification: D71, D72

Keywords: Multiparty elections, strategic voting, turnout, polarization

1 Introduction

How does political polarization affect voter turnout? Polarization might generate higher voter turnout because it increases the benefit of the favorite party or candidate winning the election relative to other parties or candidates.[1] However, when voting is strategic, voters might choose to free-ride on the turnout among their fellow voters and abstain from costly voting. Moreover, the members of the political camp whose voter base is smaller might be discouraged by the expected turnout among the opposite camp.[2] In this paper, we propose a strategic voting model to investigate the relation between electoral turnout and polarization among voters' preferences over the political candidates or parties.

We consider an electorate with a single pro-government voter group and a number of pro-opposition groups, who has to choose among one pro-government option (candidate, party or a policy proposal) and a fixed number of pro-opposition options. Our key modeling assumption is that there is ideological polarization between the pro-government camp and the

*We would like to thank the seminar participants at HSE.

†National Research University - Higher School of Economics (e-mail:kakoz@hse.ru).

‡National Research University - Higher School of Economics (e-mail:avzakharov@hse.ru).

[1] Abramowitz and Stone (2006) provide evidence from US Presidential elections of the "Bush Effect" the polarization between the two camps in the US public increased the turnout during 2004 presidential elections. Moral (2017) also documents a positive effect at a multiparty setting.

[2] Rogowski (2014) documents that divergence of policy positions during House and Senate races decrease voter turnout.

pro-opposition camp in the sense that the pro-government option is the worst for all opposition voters and the higher the polarization between the two camps, the lower the difference among the pro-opposition options for the opposition voters.

We study how the number of parties and the polarization jointly affect election results and voter turnout. We compare the cases where there is a single pro-opposition option, versus the case where there are multiple ones). This framework has been used to study costless, strategic voting on multiple occasions (see, among others, Myerson and Weber, 1993; Fey, 1997; Bouton and Castanheira, 2012; Bouton, Castanheira and Llorente-Saguer, 2017). At the same time, to our knowledge, we are the first to look at the effect of correlations among the preferences of voters on turnout when voting is costly and there are multiple parties or candidates.[3]

To illustrate our main findings suppose that there are two pro-opposition voter groups, which are denoted by A and B, represented by the opposition parties α and β. Let $a \in (0, 1)$ be the payoff received by voter group B if party α wins, and the payoff of A if β wins. Each opposition voter whose first-ranked option wins, receives the payoff 1, while receives 0 if the pro-government party wins. a captures the cohesion among the opposition against the pro-government camp. An opposition voter supporting her first-ranked option under a proportional representation system has to consider two events: her party being pivotal against the pro-government party, and being pivotal against the other opposition party. As the value of the latter pivotal vote becomes smaller, fewer of the opposition voters turn out. This result is also present under a proportional representation electoral rule, when the resulting policy is proportional to the vote shares of the pro-government party and the two pro-opposition parties.

If the two voter groups are represented only by party α, then the effect may be the opposite, because larger number of group-B voters are mobilized to vote for party α as the payoff a, the cohesion among the opposition voters, increases. This will always be the case under proportional representation, with the probability of party α winning also increasing with cohesion. Under a winner-take-all rule, the effect on the overall opposition turnout will depend on the expected number of pro-opposition voters relative to the number of pro-government voters. If the opposition is large relative to the pro-government group, then an opposite effect will be stronger — as more voters from group B mobilize to vote, the pivotal probabilities for group A voters drop, leading to a reduction in the pro-opposition, as well as the pro-government, turnout. This negative effect due to a reduction in pivotal probabilities will always be smaller than the mobilizing effect under proportional representation, where the payoff of each voter is the weighted average of her payoffs from both parties.[4]

The effect of polarization on the turnout of pro-government voters is also determined by the comparison of the sizes of the pro-government and the pro-opposition camp. When there is a single opposition party, higher opposition polarization leads to a lower pro-government turnout if the pro-government electorate is small, and to a higher turnout if the pro-government electorate is large. This result has implications for the reelection strategies of incumbents. Incumbents' choices of escalating polarization to increase the turnout among the pro-government voters could be self-defeating if the

[3]Existing literature studying costly voting and turnout largely focused on two-candidate elections with two groups of voters (Palfrey and Rosenthal, 1983, 1985; Myerson, 2000), focusing on such issues as subsidized or compulsory voting (Borgers, 2004; Krasa and Polborn, 2009), information acquisition (Tyson, 2016), polls (Goeree and Grosser, 2007), or information disclosure (Vorobyev, 2018). Another strand of literature focused on the behavior of parties or candidates with the possibility of abstention from the non-strategic voters (Adams, Dow and Merrill, 2006; Callander and Wilson, 2007).

[4]Several previous works compared costly voting in two-party elections under majoritarian and proportional systems, looking at the effects of voter asymmetry (Herrera, Morelli and Palfrey, 2014; Herrera, Morelli and Nunnari, 2016; Faravelli and Sanchez-Pages, 2015), voting cost heterogeneity, (Kartal, 2015; Krishna and Morgan, 2015), and voter informativeness (Herrera, Llorente-Saguer and McMurray, 2019) on turnout and/or aggregate welfare.

pro-government voter base is eroding against the opposition camp.[5]

Our analysis of three-party equilibria for the winner-take-all rule is limited to the symmetric case where the expected sizes of groups A and B are equal. In the generic case, we show that such equilibria do not exist when the number of voters is large; thus, two-party equilibria (shown to exist in Xefteris (2019) in a general setting) are the only ones possible.[6] This is a version of the earlier results in support of the Duverger's law (Riker, 1982; Palfrey, 1988; Fey, 1997) obtained for costless voting. In order for equilibrium to exist, a knife-edge condition has to be satisfied, because pivotal probabilities for the pro-government party and a must be comparable to the one for the pro-government party and b; that is possible only when groups A and B are of exactly the same size.

The rest of the paper is structured as follows. In Section 2 we formulate the model. In Sections 3 and 4 we analyze the equilibrium, assuming winner-take-all system and proportional representation, respectively. Section 5 concludes.

2 The model

Suppose that there are $n < \infty$ groups of citizens who need to choose from a set of $L \leq n$ policy options, each of which characterizes the policy platform of a single political party or candidate. The number of citizens is a Poisson random variable with the expected value $N > 0$; for each group $i \in \{1, \ldots, n\}$, denote by $b_i \geq 0$ the relative expected proportion of citizens of that group, with $\sum_i b_i = 1$. Citizens are distributed randomly and independently among the groups, which implies that the number of citizens in each of the groups is also a Poisson random variable with the expected value $b_i N$, and the probability that the number of type i citizens is equal to $k = 0, 1, \ldots$ is given by $e^{-b_i N}(b_i N)^k/k!$.

Let $a_{ij} \in [0, 1]$ be the payoff that a citizen from group $i \in \{1, \ldots, n\}$ receives if option j is realized, with the first-ranked option worth 1 and last-ranked option worth 0 for each citizen. Group 1 is pro-incumbent, and its first-ranked option is 1; all other options (corresponding to opposition parties or candidates) for that group are ranked last, so $a_{11} = 1$ and $a_{1j} = 0$ for $j = 2, \ldots, n$. For all pro-opposition groups of citizens option 1 is ranked last, so $a_{i1} = 0$ for $i = 2, \ldots, n$. Assume that for citizens $i = 2, \ldots, L$, we have $a_{ii} = 1$.

In our model, we concentrate on the simultaneous and strategic voting. Each citizen i must choose $s_i \in S = \{1, \ldots, L, o\}$, where the elements $1, \ldots, L$ denote policy options and o denotes abstaining. Given action profile $s \in \{1, \ldots, L, o\}^k$, let $v(s) \in \mathbb{N}^L$ be the number of votes for each choice, and let $P(v) : \mathbb{N}^L \to \Delta^L$ be the probability that each policy option is realized, given the number of votes for each option. This function depends on the electoral system or power sharing rule in place.

We consider two cases of P. Under a *winner-take-all* system, the option that receives the highest number of the votes is realized with probability one. To simplify the analysis, we make a pro-incumbent tie-breaking assumption; that is, if option 1 is tied for most votes with others, then it is realized with probability one as well. If $m > 1$ options (other than

[5] Escalation of "culture wars" by incumbent leaders has recently received some public attention. Examples range from the US (https://www.voanews.com/usa/us-politics/trump-escalates-culture-war) to Russia (https://www.washingtonpost.com/news/democracy-post/wp/2017/10/05/russias-culture-wars-look-a-lot-like-americas/) and Turkey (https://www.nytimes.com/2014/07/23/opinion/turkeys-culture-wars.html).

[6] Our approach to the analysis of equilibria in three-party elections is different from that of Arzumanyan and Polborn (2017) who assume that the voting costs are homogeneous across voters, and all voters put the same utility on their first and second-ranked alternatives; in their setting it is shown that, given a sufficiently small voting cost, there exists a mixed strategy where voters support their favorite candidates, and the candidates win with equal probability.

option 1) tie for most votes, then each is realized with probability $\frac{1}{m}$.[7] Under *proportional representation*, for each policy option $j = 1, \ldots, L$ we have $P_j(v) = \frac{v_j}{\sum_i v_i}$ if $\sum_i v_i > 0$, and $P_j = \frac{1}{L}$ otherwise.

Citizen who decides to vote pays a positive cost c that is drawn from distribution $F(\cdot)$ on $[0, 1]$. We assume that this distribution has positive density $f(\cdot)$, and that f and all higher-order derivatives of F are bounded. Let the costs of voting be independent for all citizens. Given strategy profile $s = (s_i, s_{-i})$, the payoff of citizen belonging to group i is

$$u_i(s) = \sum_{j=1}^{L} a_{ij} P_j(v(s)) - c_i \mathbf{1}_{\{s_i \neq o\}}. \tag{1}$$

The equilibrium concept we are going to use is the group-symmetric Bayesian Nash Equilibrium (BNE), where all members of the same group use the same mapping from cost values to voting behavior.

3 Winner-take-all system

For a group 1 citizen, it is a dominated action to vote for any option other than 1. However, it is not a dominant action for citizens from other groups to support one or the other opposition policy option, and, hence, the pro-opposition citizens are potentially subject to coordination motives. To isolate these effects, we separately analyze two cases in which there is a single opposition candidate or the opposition is divided in supporting one of the two policy options.

3.1 Coordinated Opposition

Suppose that all of the citizens from the opposition groups choose option 2. For any $i \in \{1, \ldots, n\}$, let π_i be probability that a randomly chosen citizen of type i votes. The expected number of votes cast for options 1 and 2, respectively, are follows:

$$V_1 = N b_1 \pi_1, \qquad V_2 = N \sum_{i=2}^{n} b_i \pi_i$$

The number of votes received by policy options 1 and 2 are also Poisson variables, with the probability of option $j = 1, 2$ receiving k votes given by

$$p_j(k) = \frac{e^{-V_j} V_j^k}{k!}.$$

The payoffs of citizen of type 1, depending on whether he votes for option 1 or abstains, are

$$u_1(s_1) = \sum_{k \geq 0} p_1(k) \sum_{i=0}^{k+1} p_2(i) - c_1, \qquad u_1(o) = \sum_{k \geq 0} p_1(k) \sum_{i=0}^{k} p_2(i).$$

Similarly, payoffs of citizen of type $j = 2, \ldots, n$ are given by

$$u_j(s_2) = a_{j2} \sum_{k \geq 0} p_2(k) \sum_{i \leq k} p_1(i) - c_j, \qquad u_j(o) = a_{j2} \sum_{k \geq 1} p_2(k) \sum_{i \leq k-1} p_1(i).$$

The equilibrium is characterized by the threshold voting costs $(c_1^*, \ldots, c_n^*)$ satisfying $u_1(s_1) = u_1(o)$, $u_i(s_2) = u_i(o)$ for $i = 2, \ldots, n$. It is immediately clear that we have $c_2^* = \frac{c_3^*}{a_{32}} = \ldots = \frac{c_n^*}{a_{n2}}$; hence, expected number of votes for options 1 and 2 will be

$$V_1^* = b_1 N F(c_1^*), \qquad V_2^* = N \left(\sum_{j=2}^{n} b_j F(a_{j2} c_2^*) \right), \tag{2}$$

[7] This tie-breaking assumption is not critical for the results.

with equilibrium (c_1^*, c_2^*) satisfying (2) and $u_1(s_1) = u_1(o)$, $u_2(s_2) = u_2(o)$.

Denote by

$$\tilde{b} = \sum_{j=2}^{n} b_j a_{j2} \tag{3}$$

the size of the opposition electorate, weighted by the strength of preferences toward the opposition policy choice.

The equilibrium should exist due to the Brouwer fixed-point theorem, and its asymptotic properties are straightforward to derive:

Proposition 1 Assume a winner-take-all system and $L = 2$. Equilibrium exists, with

$$\lim_{N\to\infty} \frac{c_1^*}{c_2^*} = \left(\frac{\tilde{b}}{b_1}\right)^{\frac{1}{3}}. \tag{4}$$

As N increases, the equilibrium threshold costs (c_1^*, c_2^*) converge to zero, but the expected number of voters for each policy option continues to grow indefinitely. Moreover, the limiting ratio of the costs (and of turnout rates) is a function of the ratio between the sizes of the pro-incumbent and opposition electorates.[8]

We now proceed to look at how polarization affects the turnout and winning probabilities when there is a single opposition policy choice. We show below that the comparative statics critically depend on the relative expected proportion of citizens types voting for option 2, weighted by the strength of their preferences. Theorem 1 below shows that the turnout rate of pro-incumbent citizens decreases with the cohesion in the opposition if and only if the relative size of the opposition is less than that of the pro-incumbent citizens.

Theorem 1 Assume a winner-take-all system and $L = 2$, and suppose that the expected number of citizens N is sufficiently large. For any opposition group $j \in \{3, \ldots, n\}$, both $\frac{\partial V_1^*}{\partial a_{j2}}$ and $\frac{\partial V_2^*}{\partial a_{j2}}$ are positive (negative) if $\tilde{b} < (>)\, b_1$.

Theorem 1 states that both pro-incumbent and pro-opposition turnout increase with the cohesion among the opposition voters if and only if the pro-opposition electorate is smaller than the pro-incumbent electorate.

There are two ways the polarization, captured by the payoff parameters $(a_{j2})_{j\geq 2}$ may affect the pivotal probabilities for the pro-incumbent voters: direct and indirect. The calculation in the proof in the Appendix show that the direct effect dominates when the electorate size is large enough. That is, when there is higher polarization, the main channel that the payoffs affect V_1^* is through their direct effect on their own cost thresholds $(c_j^*)_{j\geq 2}$. Therefore, when the size of the opposition voters, weighted by the payoff parameters $(a_{j2})_{j\geq 2}$, is higher than the size of the pro-incumbent voters, the opportunities for pro-incumbent voters to be pivotal reduce with the likelihood of voting by the opposition voters. That is, as the opposition voters' turnout increases, when the opposition is already bigger, the pivotal probabilities for the pro-incumbent voters decreases.

On the other hand, when the opposition is bigger, while pro-incumbent voters dominantly want to coordinate with each other against the opposition, which is captured by the result $\frac{\partial u_1(s_1) - u_1(o)}{\partial c_1^*} > 0$, the oppositions voters' free-riding incentives dominate. That is, as any one of the payoff parameters $(a_{j2})_{j\geq 2}$ increase while the opposition electorate is already larger, the opposition candidates enjoy the higher likelihood for a victory for opposition and can free-ride on other opposition voters.

[8] See, for example, Myerson (2000). Moreover, equilibria where most voters support two options exist for the case where there are more than two options to choose from (Xefteris, 2019).

3.2 Divided Opposition

In this section, we consider multiple policy options for which opposition citizen groups can vote. For simplicity, we will restrict our analysis to $n = L = 3$. The payoffs of a citizen from group 1 from voting for policy option 1 and from abstaining are

$$u_1(s_1) = \sum_{k\geq 0} p_1(k) \sum_{j=0}^{k+1} p_2(j) \sum_{j=0}^{k+1} p_3(j) - c_1, \qquad u_1(o) = \sum_{k\geq 0} p_1(k) \sum_{j=0}^{k} p_2(j) \sum_{j=0}^{k} p_3(j),$$

with voting for options 2 and 3 being dominated strategies. For a citizen from group 2, the payoffs from voting for options 2 and 3 and abstaining are

$$\begin{aligned}
u_2(s_2) &= \sum_{k\geq 0} p_2(k) \sum_{j=0}^{k} p_1(j) \sum_{j=0}^{k} p_3(j) + \left(\frac{1+a_{23}}{2}\right) \sum_{k\geq 1} p_2(k-1) p_3(k) \sum_{j=0}^{k-1} p_1(k) \\
&+ a_{23} \sum_{k\geq 1} p_3(k+1) \sum_{j=0}^{k} p_1(j) \sum_{j=0}^{k-1} p_2(j) - c_2, \\
u_2(s_3) &= a_{23} \sum_{k\geq 0} p_3(k) \sum_{j=0}^{k} p_1(j) \sum_{j=0}^{k} p_2(j) + \left(\frac{1+a_{23}}{2}\right) \sum_{k\geq 1} p_3(k-1) p_2(k) \sum_{j=0}^{k-1} p_1(k) \\
&+ \sum_{k\geq 1} p_2(k+1) \sum_{j=0}^{k} p_1(j) \sum_{j=0}^{k-1} p_3(j) - c_2, \\
u_2(o) &= \sum_{k\geq 0} p_2(k+1) \sum_{j=0}^{k} p_1(j) \sum_{j=0}^{k} p_3(j) + \left(\frac{1+a_{23}}{2}\right) \sum_{k\geq 1} p_2(k) p_3(k) \sum_{j=0}^{k-1} p_1(k) \\
&+ a_{23} \sum_{k\geq 1} p_3(k+1) \sum_{j=0}^{k} p_1(j) \sum_{j=0}^{k} p_2(j),
\end{aligned} \tag{5}$$

with voting for policy option 1 being a dominated strategy. Payoffs of citizens from groups 3 are defined similarly to (5).

We are interested in equilibria where each opposition group supports its first-ranked policy option. Such an equilibrium is characterized by threshold voting costs (c_1^*, c_2^*, c_3^*) satisfying $u_1(s_1) = u_1(o)$, $u_2(s_2) = u_2(o) \geq u_2(s_3)$, and $u_3(s_3) = u_3(o) \geq u_3(s_2)$, with the expected number of votes for policy option $j = 1, 2, 3$ being given by

$$V_j^* = b_j F(c_j^*). \tag{6}$$

When the electorate is large, such equilibria do exist for specific voter preferences:

Proposition 2 Assume a winner-take-all system and $L = n = 3$. An equilibrium where each voter group supports its first-ranked option or abstains exists if and only if $b_3 \frac{3-a_{32}}{2} = b_2 \frac{3-a_{23}}{2} = b_1$

Proposition 2 could be interpreted as an instance of the Duverger's Law for our model in the sense that a divided opposition with both opposition policy option have a significant presence in the elections in addition to the incumbent option is not sustainable for large electorates in general. Whenever there is an imbalance between the opposition electorate, the voters, whose favorite policy option is the disadvantaged one, find less opportunities to be pivotal in their support of their first-ranked choice. As the electorate size grows indefinitely, the likelihood of such pivotal events relative to the likelihood of the pivotal events for supporting the other opposition policy option vanishes. Reversely, sustaining three parties which have all significant vote shares requires restrictions on the preferences of the voters and the relative sizes.

As the electorate size grows indefinitely, these restrictions becomes sharper. In the limit, these restrictions reduce to the knife-edge condition stated in Proposition 2.

In the special case where the two opposition groups are of the same size and have symmetric preferences, we find that the pro-opposition turnout decreases with the cohesion cohesion between the two opposition groups.

Theorem 2 Assume a winner-take-all system, $L = n = 3$, $a_{23} = a_{32} = a$, and $b_2 = b_3$. Then $\frac{\partial V_2^*}{\partial a} = \frac{\partial V_3^*}{\partial a} < 0$ and $\frac{\partial V_1^*}{\partial a} = 0$.

As the coherence within the opposition increases, unlike the coordinated opposition case, the turnout among the pro-opposition voters reduces. This is due to the fact that the division among the opposition camp strengthens the free-rider effect among the pro-opposition voters. As a increases, so is the benefit for each opposition voter of the events that the other opposition option wins the elections. This reduces the importance of the pivotal events for the opposition voters.

3.3 Comparison of the winning probabilities under coordinated and divided opposition

The winning probabilities depend on the expected turnout rates, which in turn determine the expected number of votes. When the expected number of votes by the pro-incumbent voters is higher than that of pro-opposition voters, $V_1^* \geq V_2^*$ for the case of coordinated opposition and $V_1^* \geq \max\{V_2^*, V_3^*\}$ for the case of divided one, the likelihood that the incumbent policy option wins the election is higher than that of the opposition.[9] To understand the conditions under which this happens, we now calculate the relative turnout level for the pro-incumbent voters. We define $R_c = \frac{V_1^*}{V_1^*+V_2^*}$, where the voting rates V_1^*, V_2^* are defined in equation (2) when the opposition is coordinated. Similarly, $R_d = \frac{V_1^*}{V_1^*+V_2^*+V_3^*}$, where the voting rates are defined in (6) when the opposition is divided. The following Proposition provides the limiting values of R_c and R_d.

Proposition 3 Assume a winner-take-all system and $L = 2$. The incumbent's vote share converges in probability to

$$R_c = \frac{b_1^{\frac{2}{3}}}{b_1^{\frac{2}{3}} + \tilde{b}^{\frac{2}{3}}}. \tag{7}$$

Under a winner-take-all system, $L = n = 3$, $a_{23} = a_{32} = a$ and $b_2 = b_3$, the incumbent's vote share converges in probability to $R_d = \frac{1}{3}$.

Since $\tilde{b}$ increases with a_{j2} for all opposition groups $j = 3, \ldots, n$, the relative vote share of the incumbent, when the electorate is large, decreases with a_{j2}. When the opposition is divided, the equilibrium exists only if the vote shares are balanced. Therefore, over the set of parameters for which such an equilibrium exists, the turnout levels cancel out any change in the payoff parameters a_{j2} to keep the vote shares balanced leading to the equal and constant vote shares across the policy options. The convergence in probability follows from the property of the Poisson distribution that the expected number of votes for all policy options is on the order of $\sqrt{N}$, while the standard deviation is of the order $\sqrt[4]{N}$.

The following corollary compares the winning probability of the incumbent option for the two types of equilibria.

[9] The probability that the incumbent wins the election when the opposition is coordinated is $\sum_{k\geq 0} \rho_1(k) \sum_{j=0}^{k} \rho_2(j)$. Following Myerson (2000) we can calculate the magnitude of this probability and compare it to the probability that the opposition wins. It is possible to show that as $N \to \infty$, the probability that the incumbent option wins converges to $1(0)$ when $V_1^* > (<) V_2^*$.

Corollary 1 Assume a winner-take-all system, $n = 3$, $a_{23} = a_{32} = a$, $b_1 = b$, and $b_2 = b_3 = \frac{1-b}{2}$. The winning probability of the incumbent option converges to 1/3 when the opposition is divided, while under united opposition it converges to 1 (0) if

$$b > (<) \frac{1+a}{3+a}. \tag{8}$$

Corollary 1 provides conditions for the opposition may find beneficial to coordinate or not. The condition in equation (8) is easier to hold as increases with b and a decreases. This is as expected, since the if the size of the pro-incumbent citizens is higher, it is more likely that the incumbent option receives more votes than the opposition. On the other hand, when the coherence among the opposition increases, it is easier for the opposition to get coordinated around a single candidate. This implies that the expected benefits of coordination for the opposition leaders may depend on the underlying political preferences of the citizens. In particular, if the relative share of the pro-incumbent citizens is large enough or if there is enough difference between the ideological platforms of the opposition parties, it might make sense for the opposition leaders not to coordinate on a single party to reduce the free-riding incentives among the pro-opposition voters.

4 Proportional Representation

We next turn to the power sharing rule where the probability that each policy option is realized is proportional to the number of votes it receives (or, equivalently, a weighted policy is realized with probability one). It is straightforward to show that if one policy option is strictly preferred to all others, then it is a dominated strategy to vote for any other option, so all citizens will either support their first-ranked policy options, or abstain.

The net benefit of voting for a citizen of type 1 with cost c_1 is

$$u_1(1) - u_1(o) = \sum_{\substack{k_j \geq 0, \\ j=1,\dots,n}} \prod_{j=1}^{n} p_j(k_j) \left(\frac{k_1 + 1}{\sum_{l=1}^{n} k_l + 1} - \frac{k_1}{\sum_{l=1}^{n} k_l} \right) - c_1. \tag{9}$$

Let $\{I_2, \dots, I_L\}$ be a partition of $\{2, \dots, n\}$ such that citizen $i \in I_l$ ranks policy option l first, and let J_i be the corresponding vote of citizen i. The net benefit of voting for a type $i = 2, \dots, n$ citizen with cost c_i will be

$$u_i(s_{J_i}) - u_i(o) = \sum_{\substack{k_j \geq 0, \\ j=1,\dots,n}} \prod_{j=1}^{n} p_j(k_j) \left(\frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'} + a_{ij}}{\sum_{l=1}^{n} k_l + 1} - \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'}}{\sum_{l=1}^{n} k_l} \right) - c_i. \tag{10}$$

The equilibrium will be any $(c_1^*, \dots, c_n^*)$ satisfying (9), (10). The expected number of votes for option $1, \dots, L$ given by

$$V_1 = N b_1 F(c_1^*), \qquad V_j = N \sum_{i \in I_j} b_i F(c_i^*), j = 2, \dots, L$$

The following result is a straightforward generalization of Lemma 1 of Herrera, Morelli and Palfrey (2014) for an arbitrary number of policy options:

Lemma 1 For any $(c_1, \dots, c_n)$ we have

$$u_1(1) - u_1(o) = \frac{\sum_{j=2}^{n} b_j F(c_j)}{N(\sum_{j=1}^{n} b_j F(c_j))^2} - \frac{e^{-N(\sum_{j=1}^{n} b_j F(c_j))((\sum_{j=2}^{n} b_j F(c_j))^2 - b_1 F(c_1)^2 + N \sum_{j=2}^{n} b_j F(c_j))}}{2(\sum_{j=1}^{n} b_j F(c_j))^2} - c_1, \tag{11}$$

and

$$u_i(s_{J_i}) - u_i(o) = e^{-N\sum_{j=1}^n b_j F(c_j)} \left(\frac{N\sum_{l=2}^L a_{il} \sum_{l' \in I_l} b_{l'} F(c_{l'}) - a_{ij}}{N(\sum_{j=1}^n b_j F(c_j))^2} + \frac{\sum_{l=2}^L a_{il} \sum_{l' \in I_l} b_{l'} F(c_{l'})}{N(\sum_{l=1}^n b_l F(c_l))^2} - \frac{\sum_{l=2}^L a_{il}}{n} \right)$$
$$+ \frac{\sum_{l=1}^L (a_{ij} - a_{il}) \sum_{l' \in I_l} b_{l'} F(c_{l'})}{N(\sum_{j=1}^n b_j F(c_j))^2} - c_i, i = 2, \ldots, n. \quad (12)$$

Lemma 1 shows that, for small values of the voting costs, the net payoff from voting is a sum of an exponential and a linear terms in $(c_1, \ldots, c_n)$. The expected numbers of votes $V_1, \ldots, V_L$ diverge to infinity as $N \to \infty$, and the exponential term converges to zero faster compared with the linear terms. This allows us to obtain several asymptotic results.

4.1 Coordinated opposition

If there are only two options available, then we can derive a closed-form solution for the equilibrium:

Proposition 4 Assume proportional representation and $L = 2$. The equilibrium exists and asymptotically approaches the unique $(c_1^*, \ldots, c_n^*)$ given by

$$c_1^* = \frac{b_1^{-\frac{1}{4}} \tilde{b}^{\frac{1}{4}}}{\sqrt{f(0)N}(\sqrt{b_1} + \sqrt{\tilde{b}})}, \qquad c_j^* = \frac{a_{j2} b_1^{\frac{1}{4}} \tilde{b}^{-\frac{1}{4}}}{\sqrt{f(0)N}(\sqrt{b_1} + \sqrt{\tilde{b}})}. \quad j = 2, \ldots, n \quad (13)$$

It is straightforward to calculate the comparative statics of equilibrium with respect to citizen preferences:

Theorem 3 Assume proportional representation and $L = 2$, and suppose that the expected number of citizens N is sufficiently large. Then for any $j \in \{2, \ldots, n\}$ we have $\frac{\partial V_2^*}{\partial a_{j2}} > 0$. We have $\frac{\partial V_1^*}{\partial a_{j2}} > (<)\, 0$ if $b_1 > (<)\, \tilde{b}$.

The turnout among the opposition voters will increase under proportional representation, if the utility of any opposition voter group toward the opposition party increases. This result differs from what we find for the winner-take-all case in Theorem 1. When the benefit of voting depends on the pivotal probabilities, coordination and free-riding incentives start to play a role. Therefore, the effect of the coherence among the opposition voters on their turnout depends on the size of the opposition voter base relative to that of the pro-incumbent voters. Pivotal probabilities cease to matter for the proportional representation, hence coherence among the opposition voters directly affect the benefit of voting for opposition voters for the single opposition party.

At the same time, pro-incumbent turnout depends on opposition cohesion just as under the winner-take-all system. When the opposition is bigger in size, higher cohesion among the opposition voters increases the turnout among the opposition and therefore increases the effective gap between the opposition and the incumbent even further, which reduces the likelihood of the events that a single addition to the votes for the incumbent. When the incumbent is bigger in size, the likelihood of such events increase as the cohesion among the opposition increases, leading to a higher turnout among pro-incumbent voters.

4.2 Divided opposition

We now look at the case where there are several policy choices available to the opposition electorate. A closed-form solution similar to (13) is not possible in the generic case, so we analyze a special case where all opposition groups are of the same size, the number of opposition policy options is equal to the number of opposition voter groups, each opposition voter group has a single first-ranked option, and is indifferent between all other opposition options, and each opposition policy option is first-ranked for some voter group. Formally, we assume the following:

Assumption 1 Let $L = n$. We have $b_1 = b \in (0, 1)$ and $b_i = \frac{1-b}{n-1}$ for $i \neq 1$. We have $a_{11} = 1$ and $a_{1j} = 0$ for $j \neq 1$; for $i \neq 1$, let $a_{i1} = 0$, $a_{ii} = 1$, and $a_{ij} = a \in (0, 1)$ if $j \notin \{1, i\}$.

Here, the parameter a represents the utility that an opposition voter gets from voting for another opposition party. Equilibrium can be written out as follows:

Proposition 5 Assume proportional representation and let Assumption 1 hold. The equilibrium asymptotically approaches the unique $(c_1^*, \ldots, c_n^*)$, given by

$$c_1^* = \frac{\sqrt{(1-b)A}}{\sqrt{f(0)N}(b + (1-b)A)}, \qquad c_2^* = \ldots = c_n^* = \frac{\sqrt{1-b}A^{\frac{3}{2}}}{\sqrt{f(0)N}(b + (1-b)A)}, \tag{14}$$

where

$$A = \frac{(1-a)(n-2)}{2n-2} + \frac{1}{2}\sqrt{\frac{(1-a)^2(n-2)^2}{(n-1)^2} + \frac{4b}{1-b}}. \tag{15}$$

We derive the comparative statics of equilibrium threshold costs (14) with respect to a:

Theorem 4 Assume proportional representation, let Assumption 1 hold, and let N be sufficiently large. Then $\frac{\partial V_2^*}{\partial a} < 0$. If $b < \frac{1}{2}$, then $\frac{\partial V_1^*}{\partial a} > 0$ for all $a \in [0, 1]$. If $b \in [\frac{1}{2}, \frac{2n-3}{3n-4}]$, then $\frac{\partial V_1^*}{\partial a} > 0$ on $[0, \tilde{a}]$ and $\frac{\partial V_1^*}{\partial a} < 0$ on $[\tilde{a}, 1]$, where $\tilde{a} = \frac{(n-1)(1-2b)}{(n-2)(1-b)} + 1$. If $b > \frac{2n-4}{3n-4}$, then $\frac{\partial V_1^*}{\partial a} < 0$ for all $a \in [0, 1]$.

The decrease with a of turnout among the opposition voters is due to the decreasing marginal value of voting for one's own party, as other opposition parties become more acceptable alternatives. If all opposition parties are equally valuable to the voter, then it is worth exerting the effort only to reduce the incumbent's vote share. This is the mirror image of comparative statics for the 2-party case, where an increase in opposition cohesion mobilizes the voters from most opposition groups to vote for the united opposition party.

As with the two-party case, the effect of a on the turnout among the pro-incumbent voters depends on how many pro-opposition and pro-incumbent voters there are. If the opposition is large and pro-incumbent voters are a minority, then an increase in a will lower the pro-opposition turnout and increase the marginal value of voting for the pro-incumbents. If there are relatively few pro-opposition voters, then lower pro-opposition turnout will increase the incumbent party vote share by so much that the pro-incumbent voters will be demobilized. Thus the effect of opposition cohesion on turnout among pro-incumbent voters in the case of divided opposition will also be the opposite of that effect for the united opposition.

4.3 Comparison of turnout under coordinated and divided opposition

The policy outcome under proportional representation is determined by the share of pro-incumbent vote relative to the total vote. We next examine how this is affected by the number of opposition options available. The relative shares are given by the following statement.

Proposition 6 Assume proportional representation and $L = 2$. The incumbent's relative vote share converges in probability to

$$R_c = \frac{\sqrt{b_1}}{\sqrt{b_1} + \sqrt{\tilde{b}}}. \tag{16}$$

Under proportional representation and Assumption 1, incumbent's relative vote share converges in probability to

$$R_d = \frac{b}{b + (1-b)A}. \tag{17}$$

The value R_c will decrease with any a_{j2}, so any increase in cohesion among pro-opposition voter groups will produce an increase in the relative vote share of the united opposition. At the same time, R_d will increase in a, so an increase in opposition cohesion will increase the pro-incumbent relative vote share due to the free-riding of the voters from the various opposition camps.

If the preferences of the citizens are given as in Assumption 1, then we have

$$b_1 = b, \qquad \tilde{b} = \frac{(1-b)(1-2a+an)}{n-1}. \tag{18}$$

We can then compare R_d and R_c under conditions (18):

Corollary 2 Under (18), we have $R_c > R_d$ if $a \in [0, 1)$ and $R_c = R_d$ if $a = 1$.

Facing a united opposition always produces a more favorable result for the incumbent, compared with facing divided opposition. The exception is when all opposition vote options are perfect substitutes for all groups of opposition voters. In that case, both under divided and united opposition the voters only care about increasing the vote share of their favorite option relative to the incumbent. If the cohesion among opposition voter groups is less than perfect, then, in case of united opposition, some of the voters will be dissuaded by having to vote for a policy option that is not their first-ranked. If the opposition is divided, then each voter will vote for her favorite option competing against other opposition options as well as the incumbent, which will further increase the turnout.

5 Discussion

The causes of political polarization among voters[10] range from the behavioral ones (Ortoleva and Snowberg, 2015) to candidates' equilibrium policy divergence (Callander and Wilson, 2007). In this paper, we focus on the electoral

[10]The US public is increasingly polarized along the political lines. Even though there is some evidence suggesting that the mass public is less polarized than the elites(As documented by DiMaggio, Evans and Bryson, 1996; Fiorina and Abrams, 2008, for example); some recent studies document political polarization among the US public is prevalent and still is on the rise (see Abramowitz and Saunders, 2008; McCright and Dunlap, 2011; Druckman, Peterson and Slothuus, 2013; Iyengar and Westwood, 2015, among others). Mason (2015) argues that polarization is a result of better sorting of the mass public along the partisan lines, hence in a sense caused by higher levels of political participation.

consequences of polarization in a multi-party or multi-candidate elections by providing a theoretical analysis using the costly strategic voting framework . Our results provide a basis for forming testable hypotheses regarding the impacts of changes in voters preferences on turnout ratios and other observable outcomes. We show that the turnout among the pro-incumbent voters decrease due to polarization if the pro-incumbent voters are less populated than the pro-opposition voters. This result holds for both winner-take-all type electoral system and proportional representation. The turnout in the opposition camp, on the other hand, increases with polarization for proportional systems, while it can decrease with it for a winner-take-all system if the pro-incumbent voter base is larger.

One of the under-explored dimensions of political polarization is its relation with "problem of divided majority;" (Bouton and Castanheira, 2012) in particular, the coordination problem of a divided opposition against a long-standing incumbent. The leaders of opposition parties often face the challenge of coordinating the opposition behind a single front against an incumbent. Polarization between the pro-incumbent and the pro-opposition camps might help the opposition leaders to unify the pro-opposition voter along their discontent about the incumbent camp. On the other hand, if the ideological differences are large within the opposition camp, coordinating on one of the opposition parties might reduce the motivation of the voters from other opposition parties. In addition to the free-riding incentives among the pro-opposition voters might prevent the benefits of a unified opposition camp from realizing.

Let's consider an illustrative example from the electoral politics in Turkey. The conservative ruling party in Turkey, AKP, holds the central government since 2002, and the mayoral offices of Ankara and Istanbul, the two largest cities of Turkey since 1994. One of the results of such a long period of incumbency is the widening polarization between the pro-AKP electorate and the pro-opposition one in Turkey.[11] There are four major parties in the opposition camp: the center-left and the secular-nationalist CHP, the leftist and pro-Kurdish HDP, right-wing nationalist MHP, center-right and secular nationalist IYIP[12]. Before the 2018 presidential elections AKP and MHP formed an alliance to support the presidency of Erdogan, the leader of AKP and current president. As a response, CHP and IYIP formed another alliance to support the presidential candidate of CHP. HDP, even though its vote base is larger than MHP and IYIP, was not included in the coalition between CHP and IYIP. One of the factors behind this is the ideological differences between the HDP's mostly Kurdish vote base and CHP's more nationalist vote base. The increasing polarization between the pro-incumbent and the opposition camps did not help the opposition during the 2018 presidential elections; Erdogan won by getting 52.59% of the votes. However, the candidates supported by CHP and IYIP during the 2019 mayoral elections in Ankara (with a vote share of 50.93%) and Istanbul(with a vote share of 48.80%) won the elections.[13] our analysis could provide some guidance in interpreting the results. Our theoretical analysis in this paper predict that polarization could increase the likelihood of a victory by the incumbent if the pro-opposition camp is smaller than the pro-incumbent camp (as it was arguably the case in Turkey during the 2018 presidential elections) or if the electoral system is proportional. And in those cases it might make sense for the opposition to not unify the opposition and emphasize the differences among each other. However, when the opposition's vote base is larger (as it was the case in Ankara and Istanbul during

[11] See `https://konda.com.tr/wp-content/uploads/2019/05/1904Nisan_Barometre_97_Kamuoyu.pdf` for a report based on survey results collected after the 2019 local elections. One of the main conclusions of the report is that there is a significant polarization among the Turkish electorate during the 2019 local elections and the division is mainly along the pro-incumbent vs. opposition camps (p13).

[12] IYIP was founded in 2017 mostly by former members of MHP as a response to the increasing cooperation between MHP and the incumbent party AKP.

[13] HDP also implicitly supported these candidates by not nominating any candidates in these two cities.

the 2019 mayoral elections), coordination within the opposition camp might enable them to reap the benefits of higher polarization. This could help explaining the failure of coordination between CHP-IYIP and HDP during the legislative and presidential elections while HDP's support for the CHP's candidates in Ankara and Istanbul during the 2019 mayoral elections. On the other hand, there are many factors in play for these results, and a rigorous analysis that takes the dynamic interaction between the incumbent, the opposition leaders and the strategic voters remains to be an interesting open problem, and is beyond the scope of this paper.

A Proofs

Proof of Proposition 1 Put

$$H_1 = u_1(s_1) - u_1(o) = \sum_{k\geq 0} p_1(k)p_2(k+1) - c_1, \qquad H_2 = u_2(s_2) - u_2(o) = \sum_{k\geq 0} p_1(k)p_2(k) - c_2. \tag{19}$$

The equilibrium condition $H_1 = H_2 = 0$ have a solution because the mapping $(c_1, c_2) \mapsto (H_1+c_1, H_2+c_2)$ is continuous and from $[0,1]^2$ into itself, so by Brouwer's fixed point theorem there exists a fixed point $(c_1^*, c_2^*) \in [0,1]^2$ where we will have $H_1 = H_2 = 0$. Moreover, both c_1^* and c_2^* are strictly between 0 and 1. Since payoffs from candidates are bounded by 1, and the pivotal probabilities are strictly lower than 1, $c_1^*, c_2^* < 1$. On the other hand, note that $c_2^* > 0$, since the pivotal probability in the equilibrium condition $H_2 = 0$ remains positive even if $c_2^* = 0$. Therefore, the pivotal probability in the equilibrium condition $H_1 = 0$ remains positive when $c_1^* = 0$ and $c_2^* > 0$. This implies that $c_1^*, c_2^* > 0$.

Rewrite H_1 and H_0 using modified Bessel functions of the first kind by applying the formula 9.6.10 in Abramowitz and Stegun (1948):

$$H_1 = e^{-V_1-V_2}\frac{\sqrt{V_2}}{\sqrt{V_1}} I_1(2\sqrt{V_1V_2}) - c_1, \qquad H_2 = e^{-V_1-V_2} I_0(2\sqrt{V_1V_2}) - c_2. \tag{20}$$

As $\lim_{N\to\infty} V_1V_2 = \infty$ is large, we can use the approximation using the formulas 9.7.1 and 9.7.7 in Abramowitz and Stegun (1948):

$$H_1 = e^{-V_1-V_2}\frac{\sqrt{V_2}}{\sqrt{V_1}} \frac{e^{\sqrt{4V_1V_2+1}}}{\sqrt{2\pi\sqrt{4V_1V_2+1}}} - c_1, \tag{21}$$

$$H_2 = e^{-(\sqrt{V_1}-\sqrt{V_2})^2} \frac{1}{\sqrt{4\pi\sqrt{V_1V_2}}} - c_2. \tag{22}$$

Set $H_1 = H_2 = 0$. Since the higher-order derivatives of $F(\cdot)$ at zero are bounded, by approximating F near zero using Taylor's Theorem we can write

$$\frac{V_2}{V_1} = \frac{\sum_{j=2}^n b_j F(a_{j2}c_2^*)}{b_1 F(c_1^*)} \simeq \frac{c_2^*\tilde{b}}{c_1^* b_1}$$

and hence we have

$$\lim_{N\to\infty}\frac{c_1^*}{c_2^*} = \lim_{N\to\infty}\sqrt{\frac{c_2^*\tilde{b}}{c_1^*b_1}} = \left(\frac{\tilde{b}}{b_1}\right)^{\frac{1}{3}}. \tag{23}$$

Q.E.D.

Proof of Theorem 1.

Differentiating H_1 and H_2 with respect to the cost thresholds yield after some algebraic reorganization

$$\begin{aligned}
\frac{\partial H_1}{\partial c_1} &= b_1 N f(c_1^*)(A_1 - A_2) - 1, \qquad \frac{\partial H_1}{\partial c_2} = N\left(b_2 f(c_2) + \sum_{j=3}^n b_j a_{j2} f(a_{j2}c_2)\right)(A_3 - A_2),\\
\frac{\partial H_2}{\partial c_1} &= b_1 N f(c_1^*)(A_2 - A_3), \qquad \frac{\partial H_2}{\partial c_2} = N\left(b_2 f(c_2) + \sum_{j=3}^n b_j a_{j2} f(a_{j2}c_2)\right)(A_4 - A_3) - 1,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
A_1 &= \sum_{k=0}^{\infty} p_1(k)p_2(k+2), \qquad A_2 = \sum_{k=0}^{\infty} p_1(k)p_2(k+1),\\
A_3 &= \sum_{k=0}^{\infty} p_1(k)p_2(k), \qquad A_4 = \sum_{k=1}^{\infty} p_1(k)p_2(k-1).
\end{aligned}$$

We can express these terms in terms of modified Bessel functions as follows:

$$\begin{aligned} A_1 &= \frac{V_2}{V_1} e^{-V_1-V_2} I_2(2\sqrt{V_1V_2}), \qquad A_2 = \frac{\sqrt{V_2}}{\sqrt{V_1}} e^{-V_1-V_2} I_1(2\sqrt{V_1V_2}), \\ A_3 &= e^{-V_1-V_2} I_0(2\sqrt{V_1V_2}), \qquad A_4 = \frac{\sqrt{V_1}}{\sqrt{V_2}} e^{-V_1-V_2} I_1(2\sqrt{V_1V_2}). \end{aligned}$$

From (2) it follows that

$$N = \frac{V_1}{b_1F(c_1)} = \frac{V_2}{\sum_{j=2}^n b_jF(a_{j2}c_2)}.$$

Substitution these expressions into (24), and imposing the equilibrium conditions that $H_1 = H_2 = 0$, we get

$$\begin{aligned} \frac{\partial H_1}{\partial c_1^*} &= G_1V_1\left(\frac{\sqrt{V_2}}{\sqrt{V_1}}\frac{I_2(2\sqrt{V_1V_2})}{I_1(2\sqrt{V_1V_2})} - 1\right) - 1, \qquad \frac{\partial H_1}{\partial c_2^*} = G_2V_2\left(1 - \frac{\sqrt{V_2}}{\sqrt{V_1}}\frac{I_1(2\sqrt{V_1V_2})}{I_0(2\sqrt{V_1V_2})}\right), \\ \frac{\partial H_2}{\partial c_1^*} &= -G_1V_1\left(\frac{\sqrt{V_1}}{\sqrt{V_2}}\frac{I_0(2\sqrt{V_1V_2})}{I_1(2\sqrt{V_1V_2})} - 1\right), \qquad \frac{\partial H_2}{\partial c_2^*} = G_2V_2\left(\frac{\sqrt{V_1}}{\sqrt{V_2}}\frac{I_1(2\sqrt{V_1V_2})}{I_0(2\sqrt{V_1V_2})} - 1\right) - 1. \end{aligned} \tag{25}$$

where

$$G_1 = \frac{c_1^* f(c_1^*)}{F(c_1^*)}, \qquad G_2 = \frac{c_2^*\left(\sum_{j=2}^n b_j a_{j2} f(a_j c_2^*)\right)}{\left(\sum_{j=2}^n b_j F(a_{j2}c_2^*)\right)}.$$

As

$$\lim_{N\to\infty} V_1V_2 = \infty \text{ and } \lim_{N\to\infty} c_1^* = \lim_{N\to\infty} c_2^* = 0.$$

It follows that $\lim_{N\to\infty} G_1 = \lim_{N\to\infty} G_2 = 1$, by Taylor's Theorem. Moreover, by applying the formula 9.7.1 in Abramowitz and Stegun (1964) one can show that $\lim_{x\to\infty}\frac{I_i(x)}{I_j(x)} = 1$ for $i, j \geq 0$.

Now let's define the following function

$$\begin{aligned} D(g_1, g_2, v_1, v_2, i_0, i_1, i_2) = {} & \left(g_1v_1\left(\frac{\sqrt{v_2}}{\sqrt{v_1}}\frac{i_2}{i_1} - 1\right) - 1\right)\left(g_2v_2\left(\frac{\sqrt{v_1}}{\sqrt{v_2}}\frac{i_1}{i_0} - 1\right) - 1\right) \\ & + g_1g_2v_1v_2\left(1 - \frac{\sqrt{v_2}}{\sqrt{v_1}}\frac{i_1}{i_0}\right)\left(\frac{\sqrt{v_1}}{\sqrt{v_2}}\frac{i_0}{i_1} - 1\right). \end{aligned}$$

Then

$$D(G_1, G_2, V_1, V_2, I_0(2\sqrt{V_1V_2}), I_1(2\sqrt{V_1V_2}), I_2(2\sqrt{V_1V_2})) = det\left(\begin{bmatrix} \frac{\partial H_1}{\partial c_1}(c_1^*, c_2^*) & \frac{\partial H_1}{\partial c_2}(c_1^*, c_2^*) \\ \frac{\partial H_2}{\partial c_1}(c_1^*, c_2^*) & \frac{\partial H_2}{\partial c_2}(c_1^*, c_2^*) \end{bmatrix}\right)$$

is the determinant of the Jacobian matrix for the net benefits to voting evaluated at equilibrium values of c_1^* and c_2^*. On the other hand, for any $I \neq 0$, $V > 0$, and $K > 0$, we have

$$D(1, 1, V, \frac{V}{K^2}, I, I, I) = \frac{V}{K^2} - 2\frac{V}{K} + V + 1 = V\frac{(K-1)^2}{K^2} + 1, \tag{26}$$

which is positive whenever $K > 0$. Taking $K = \lim_{N\to\infty}\sqrt{\frac{V_2}{V_1}}$, we have

$$\lim_{N\to\infty}\frac{D(G_1, G_2, V_1, V_2, I_0(2\sqrt{V_1V_2}), I_1(2\sqrt{V_1V_2}), I_2(2\sqrt{V_1V_2}))}{D(1, 1, V, \frac{V}{K^2}K^4, I, I, I)} = 1,$$

which also implies that the determinant of the Jacobian matrix for the net benefits to voting evaluated at equilibrium values of c_1^* and c_2^* is always positive when the expected number of citizens N is large enough.

We now compute the partial derivatives of H_1 and H_2 with respect to a_{j2} for $j = 3, \ldots n$:

$$\begin{aligned} \frac{\partial H_1}{\partial a_{j2}} &= b_jNc_2f(a_{j2}c_2)(A_3 - A_2) = \frac{\partial H_1}{\partial c_2}\frac{c_2^* b_j f(a_{j2}c_2^*)}{\sum_{k=2}^n b_k a_{k2} f(a_{k2}c_2)}, \\ \frac{\partial H_2}{\partial a_{j2}} &= b_jNc_2f(a_{j2}c_2)(A_4 - A_3) = \left(\frac{\partial H_2}{\partial c_2} + 1\right)\frac{c_2 b_j f(a_{j2}c_2)}{\sum_{k=2}^n b_k a_{k2} f(aa_{k2}c_2)}. \end{aligned} \tag{27}$$

The implicit function theorem gives us

$$\frac{\partial c_1^*}{\partial a_{j2}} = \frac{b_j c_2^* f(a_{j2}c_2^*)}{D(\sum_{k=2}^n b_k a_{k2} f(a_{k2}c_2^*))} \frac{\partial H_1}{\partial c_2^*}. \tag{28}$$

When N is sufficiently large, the determinant D is positive, so the sign of $\frac{\partial c_1^*}{\partial a_j}$ is identical to that of $\frac{\partial H_1}{\partial c_2^*}$ evaluated at equilibrium. The sign of that derivative, in turn, is identical to that of $1 - \frac{\sqrt{V_2}}{\sqrt{V_1}} \frac{I_1(2\sqrt{V_1 V_2})}{I_0(2\sqrt{V_1 V_2})}$, the latter value converges so $1 - \frac{\tilde{b}}{b_1}$ as $N \to \infty$.

The comparative statics for c_1^* follows immediately given the calculations above: If $\tilde{b} > (<) b_1$, then c_1^* is decreasing (increasing) in a_{j2}.

Now, let's look at the comparative statics for c_2^*. By the implicit function theorem, we have

$$\frac{\partial c_2^*}{\partial a_{j2}} = -\frac{b_j c_2^* f(a_{j2}c_2^*)}{\sum_{k=2}^n b_k a_{k2} f(a_{k2}c_2^*)} \frac{D + \frac{\partial H_1}{\partial c_1^*}}{D}. \tag{29}$$

We can compute $\frac{\partial \sum_k \pi_k^*}{\partial a_{j2}}$ as follows

$$\begin{aligned} \frac{\partial \sum_k \pi_k^*}{\partial a_{j2}} &= \frac{\partial c_2^*}{\partial a_{j2}} \left(\sum_{k=2}^n b_k a_{k2} f(a_{k2}c_2^*) \right) + b_j c_2^* f(a_{j2}c_2^*) \\ = -b_j c_2^* f(a_{j2}c_2^*) \frac{D + \frac{\partial H_1}{\partial c_1^*}}{D} &+ b_j c_2^* f(a_{j2}c_2^*) = -\frac{b_j c_2^* f(a_{j2}c_2^*)}{D} \frac{\partial H_1}{\partial c_1^*}. \end{aligned}$$

The determinant D is positive if N is sufficiently large. The derivative $\frac{\partial H_1}{\partial c_1^*}$ is positive if $\tilde{b} > b_1$ and is negative if $\tilde{b} < b_1$, for a large enough N. **Q.E.D.**

Proof of Proposition 2

Step 1. The sum of the expected vote shares $V_1^* + V_2^* + V_3^*$ diverges to ∞ as $N \to \infty$.

First note that if V_1^* is bounded, then c_1^* converges to 0 as $N \to \infty$. Now suppose that there is a subsequence of V_1^* that diverges to infinity. Then the expected net benefit of supporting the candidate 1 for citizens of group 1 from the indifference of the first group of voters can be bounded as

$$\begin{aligned} &\sum_{k \geq 0} p_1(k) \left(p_2(k+1) \sum_{j=0}^{k+1} p_3(j) + p_3(k+1) \sum_{j=0}^{k} p_2(k) \right) \\ &\leq p_1(V_1^*) \sum_{k \geq 0} \left(p_2(k+1) \sum_{j=0}^{k+1} p_3(j) + p_3(k+1) \sum_{j=0}^{k} p_2(k) \right), \end{aligned}$$

since the voting probability distribution function $p_i(k)$ receives its maximum value at $p_i(V_i)$. Without loss of generality, assume that V_1^* is integer-valued. Since $p_1(V_1^*) = \frac{e^{-V_1^*}(V_1^*)^{V_1^*}}{(V_1^*)!}$, we can use the Stirling's Formula to obtain

$$\lim_{N \to \infty} p_1(V_1^*) = \lim_{N \to \infty} \frac{e^{-V_1^*}(V_1^*)^{V_1^*}}{e^{-V_1^*}(V_1^*)^{V_1^*}\sqrt{2\pi V_1^*}} = 0,$$

which implies that c_1^* converges to 0. A similar argument can be used to show that all of the cost thresholds must converge to 0. Then, the following term converges to zero as well

$$p_1(0)p_2(0)p_3(0) = e^{-V_1^* - V_2^* - V_3^*},$$

and therefore the sum of the voting rates $V_1^* + V_2^* + V_3^*$ diverge to ∞.

Step 2. Let $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. If the cost-threshold ratios c_{i_1}/c_{i_2}, c_{i_1}/c_{i_3} converge to finite γ_{12}, γ_{13}, then we have

$$\lim_{N\to\infty} \frac{\sum_{k=1} p_{i_1}(k) p_{i_2}(k) \sum_{j=0}^{k} p_{i_3}(j)}{\sum_{k=1} p_{i_1'}(k) p_{i_2'}(k) \sum_{j=0}^{k} p_{i_3'}(j)} = \infty\ (0)$$

if $\mu_{i_1 i_2 i_3} > (<)\ \mu_{i_1' i_2' i_3'}$, where

$$\mu_{i_1 i_2, i_3} = \begin{cases} -\frac{b_{i_1}}{b_1}\gamma_{i_1 1} - \frac{b_{i_2}}{b_1}\gamma_{i_2 1} + 2\frac{\sqrt{b_{i_1}\gamma_{i_1 1} b_{i_2}\gamma_{i_2 1}}}{b_1}, & \text{if} \quad b_{i_3}\gamma_{i_3 1} \le \sqrt{b_{i_1}\gamma_{i_1 1} b_{i_2}\gamma_{i_2 1}} \\ -\frac{b_{i_1}}{b_1}\gamma_{i_1 1} - \frac{b_{i_2}}{b_1}\gamma_{i_2 1} - \frac{b_{i_3}}{b_1}\gamma_{i_3 1} + 3\frac{(b_{i_1}\gamma_{i_1 1} b_{i_2}\gamma_{i_2 1} b_{i_3}\gamma_{i_3 1})^{\frac{1}{3}}}{b_1}, & \text{if} \quad b_{i_3}\gamma_{i_3 1} > \sqrt{b_{i_1}\gamma_{i_1 1} b_{i_2}\gamma_{i_2 1}}. \end{cases} \tag{30}$$

First, note that since $\lim_{N\to\infty} V_1^* + V_2^* + V_3^* = \infty$ as we proved in Step 1, the limiting ratios of the cost thresholds imply $\min\{V_1^*, V_2^*, V_3^*\} \to \infty$ as $N \to \infty$. This enables us to calculate the "magnitude" of all events which are part of the pivotal events for the voters, following Myerson (2000).

Let E be an event, and let $P^*(E)$ denote the probability of this event given the equilibrium strategies. Then the magnitude of $P^*(E)$ is defined as $\lim_{N\to\infty} \frac{ln(P^*(E))}{V_1^*}$; that is we relativize all events with respect to the equilibrium number of votes that candidate 1 receives.

By Theorem 1 in Myerson (2000) the probability masses accumulate almost surely around the maximum point of each event. Therefore,

$$\begin{aligned} \mu_{i_1 i_2, i_3} &= \lim_{N\to\infty} \ln\left(\sum_{k=1}^{\infty} p_{i_1}(k) p_{i_2}(k) \sum_{j=0}^{k} p_{i_3}(j)\right) \frac{1}{V_1^*} = \\ &= \lim_{N\to\infty} \max_{k,j\le k} \left(-V_{i_1}^* - V_{i_2}^* - V_{i_3}^* - k\ln\frac{k^2}{V_{i_1}^* V_{i_2}^*} - j\ln\frac{j}{V_{i_3}^*} + 2k + j\right)\frac{1}{V_1^*}. \end{aligned}$$

Maximization w.r.t k is equivalent to maximizing $-2k\ln k + k\ln(V_{i_1}^* V_{i_2}^*) + 2k$, which leads to the solution $k = \sqrt{V_{i_1}^* V_{i_2}^*}$. Maximization w.r.t. j is equivalent to maximizing $-j\ln j + j\ln V_{i_3}^* + j$ which leads to the solution $j = \min\{V_{i_3}^*, \sqrt{V_{i_1}^* V_{i_2}^*}\}$. Substituting these values and the definitions of $V_{i_j}^* = b_{i_j} c_{i_j}^* N$ leads to expression (30). The statement follows from Corollary 1 of Myerson (2000), which implies that the limiting ratio of two events is 0 if and only if the magnitude of event on the denominator is greater than the magnitude of event on the nominator.

Step 3. Suppose that there is a subsequence of (c_1^*, c_2^*, c_3^*) such that $\lim_{N\to\infty} \frac{c_1^*}{c_2^*} = \gamma_{12} = \frac{1}{\gamma_{21}} \in (0,1)$ and $\lim_{N\to\infty} \frac{c_1^*}{c_3^*} = \gamma_{13} = \frac{1}{\gamma_{31}} \in (0,1)$. Then $b_3\frac{3-a_{32}}{2} = b_2\frac{3-a_{23}}{2} = b_1$.

In equilibrium, the following conditions must be satisfied:

$$\begin{aligned} H_{1d} &= \sum_{k\ge0} p_1(k)\left(p_2(k+1)\sum_{j=0}^{k+1} p_3(j) + p_3(k+1)\sum_{j=0}^{k} p_2(j)\right) - c_1^* = 0, \\ H_{2d} &= \sum_{k\ge0} p_1(k)p_2(k)\sum_{j=0}^{k} p_3(j) + \left(\frac{1-a_{23}}{2}\right)\sum_{k\ge1} p_3(k)(p_2(k)+p_2(k-1))\sum_{j=0}^{k-1} p_1(j) - c_2^* = 0, \\ H_{3d} &= \sum_{k\ge0} p_1(k)p_3(k)\sum_{j=0}^{k} p_2(j) + \left(\frac{1-a_{32}}{2}\right)\sum_{k\ge1} p_2(k)(p_3(k)+p_3(k-1))\sum_{j=0}^{k-1} p_1(j) - c_3^* = 0. \end{aligned} \tag{31}$$

Before proceeding, we define the following indifference conditions for the citizens in the opposition groups between deviations to support the other opposition candidate and not voting. The difference for citizens of group 3 to deviate to support the candidate 2 and not voting is

$$H_{32d} = a_{32} \sum_{k \geq 0} p_1(k) p_2(k) \sum_{j=0}^{k} p_3(j) - \left(\frac{1 - a_{32}}{2} \right) \sum_{k \geq 1} p_3(k)(p_2(k) + p_2(k-1)) \sum_{j=0}^{k-1} p_1(j) \tag{32}$$

and let's define $c_{32d} = H_{32d}$ as the maximum cost that makes them indifferent between voting and not. The indifference for citizens of group 2 to deviate to support the candidate 3 and not voting is

$$H_{23d} = a_{23} \sum_{k \geq 0} p_1(k) p_3(k) \sum_{j=0}^{k} p_2(j) - \left(\frac{1 - a_{23}}{2} \right) \sum_{k \geq 1} p_2(k)(p_3(k) + p_3(k-1)) \sum_{j=0}^{k-1} p_1(j), \tag{33}$$

where $c_{23d} = H_{23d}$ is defined as before. Note that $c_3^* > c_{32}^* \Leftrightarrow$ group 3 citizens would not prefer to deviate to supporting the candidate 2.

Case 1.1: $b_1 > b_2\gamma_{21} > b_3\gamma_{31}$.

Case 1.1.1: $b_2\gamma_{21} > \sqrt{b_1 b_3 \gamma_{31}}$.

Consider the limit $\lim_{N\to\infty} \frac{c_{32d}}{c_3^*}$. The magnitude of $\mu_{12,3} = -1 - \frac{b_2}{b_1}\gamma_{21} + 2\frac{\sqrt{b_2}\gamma_{21}}{b_1}$, which is bigger than both $\mu_{13,2}$ and $\mu_{23,1}$, which equal to $-1 - \frac{b_2}{b_1}\gamma_{21} - \frac{b_3}{b_1}\gamma_{31} + 3\frac{(b_2\gamma_{21}b_3\gamma_{31})^{\frac{1}{3}}}{b_1}$. Therefore, the magnitude of c_{32d} is $\mu_{12,3} + \left(\frac{1-a_{32}}{2}\right)\mu_{23,1}$, while the magnitude of c_3^* is $\mu_{13,2} + \left(\frac{1-a_{32}}{2}\right)\mu_{23,1}$, which is smaller than that of c_{32}^*. Hence, for large enough electorate, citizens of group 3 would benefit from deviating to voting for candidate 2.

Case 1.1.2: $b_2\gamma_{21} \leq \sqrt{b_1 b_3 \gamma_{31}}$. Consider the limit $\lim_{N\to\infty} \frac{c_1^*}{c_3^*}$. The magnitude of $\mu_{12,3} = -1 - \frac{b_2}{b_1}\gamma_{21} + 2\frac{\sqrt{b_2}\gamma_{21}}{b_1}$, which is bigger than $\mu_{23,1} = -1 - \frac{b_2}{b_1}\gamma_{21} - \frac{b_3}{b_1}\gamma_{31} + 3\frac{(b_2\gamma_{21}b_3\gamma_{31})^{\frac{1}{3}}}{b_1}$. Since the other term is common in both threshold definitions, $\lim_{N\to\infty} \frac{c_1^*}{c_3^*} = \infty$, which is a contradiction with the supposition that γ_{31} is finite.

Case 1.2: $b_1 > b_3\gamma_{31} > b_2\gamma_{21}$.

This case is symmetric to Case 1 above, where we exchange candidate 3 with candidate 2.

Case 1.3: $b_1 > b_3\gamma_{31} = b_2\gamma_{21} > 0$.

In this case, we have $\mu_{12,3} = \mu_{13,2} > \mu_{23,1}$, which implies that $\lim_{N\to\infty} \frac{c_1^*}{c_3^*} = \infty$.

Case 2.1: $b_2\gamma_{21} > b_1 \geq b_3\gamma_{31}$.

In this case, $\mu_{12,3} > \mu_{13,2}$, which implies $\lim_{N\to\infty} \frac{c_{32d}}{c_3^*} = \infty$ and leads group 3 voters to deviate to support candidate 2.

Case 3.1: $b_3\gamma_{31} > b_1 \geq b_2\gamma_{21}$.

In this case, $\mu_{13,2} > \mu_{12,3}$, which implies $\lim_{N\to\infty} \frac{c_{23d}}{c_2^*} = \infty$ and leads group 2 voters to deviate to support candidate 3.

Case 4: $b_3\gamma_{31} = b_2\gamma_{21} > b_1$.

In this case, $\mu_{23,1} = -\frac{b_2}{b_1}\gamma_{21} - \frac{b_3}{b_1}\gamma_{31} + 2\frac{\sqrt{b_2\gamma_{21}b_3\gamma_{31}}}{b_1} > \mu_{12,3} = \mu_{13,2}$. This implies that $\lim_{N\to\infty} \frac{c_2^*}{c_1^*} = \infty$, which is a contradiction.

Case 5: $b_3\gamma_{31} = b_2\gamma_{21} = b_1$.

In this case, $\mu_{23,1} = \mu_{12,3} = \mu_{13,2}$. Therefore, $\gamma_{23} = \frac{3-a_{23}}{3-a_{32}}$, while $\gamma_{12} = \frac{2}{3-a_{23}}$, and $\gamma_{13} = \frac{2}{3-a_{32}}$. Then, $b_3\gamma_{31} = b_2\gamma_{21} = b_1$ implies

$$b_3 \frac{3 - a_{32}}{2} = b_2 \frac{3 - a_{23}}{2} = b_1.$$

Q.E.D.

Proof of Theorem 2. Differentiating the equilibrium indifference conditions with respect to c_1, c_2, and c_3 yields

$$\begin{aligned}
\frac{H_{1d}}{\partial c_1} &= N_1 \sum_{k\geq 1}(p_1(k-1)-p_1(k))\left(p_2(k+1)\sum_{j=0}^{k+1}p_3(j)+p_3(k+1)\sum_{j=0}^{k}p_2(k)\right) \\
&- N_1 p_1(0)(p_2(0)p_3(1)+p_2(1)p_3(0)+p_2(1)p_3(1))-1, && (34)\\
\frac{H_{1d}}{\partial c_2} &= N_2 \sum_{k\geq 0}p_1(k)\left((p_2(k)-p_2(k+1))\sum_{j=0}^{k+1}p_3(j)-p_2(k)p_3(k+1)\right)-N_2, && (35)\\
\frac{H_{1d}}{\partial c_3} &= N_3 \sum_{k\geq 0}p_1(k)\left((p_3(k)-p_3(k+1))\sum_{j=0}^{k+1}p_2(j)-p_3(k)p_2(k+1)\right), && (36)
\end{aligned}$$

$$\begin{aligned}
\frac{H_{2d}}{\partial c_1} &= N_1 \sum_{k\geq 1}(p_1(k-1)-p_1(k))p_2(k)\sum_{j=0}^{k}p_3(j) - N_1\frac{1-a}{2}\sum_{k\geq 1}p_3(k)(p_2(k)+p_2(k-1))p_1(k-1) - \\
&- N_1 p_1(0)p_2(0)p_3(0), && (37)\\
\frac{H_{2d}}{\partial c_2} &= N_2 \sum_{k\geq 1}p_1(k)(p_2(k-1)-p_2(k))\sum_{j=0}^{k}p_3(j) - N_2\frac{1-a}{2}\sum_{k\geq 2}p_3(k)(p_2(k-2)-p_2(k))\sum_{j=0}^{k-1}p_1(k) - \\
&- N_2 p_1(0)p_2(0)p_3(0) - N_2\frac{1-a}{2}p_1(0)p_2(1)p_3(1)-1, && (38)\\
\frac{H_{2d}}{\partial c_3} &= N_3 \sum_{k\geq 0}p_1(k)p_2(k)p_3(k) + N_3\frac{1-a}{2}\sum_{k\geq 1}(p_3(k-1)-p_3(k))(p_2(k)+p_2(k+1))\sum_{j=0}^{k-1}p_1(k), && (39)
\end{aligned}$$

$$\begin{aligned}
\frac{H_{3d}}{\partial c_1} &= N_1 \sum_{k\geq 1}(p_1(k-1)-p_1(k))p_3(k)\sum_{j=0}^{k}p_2(j) - N_1\frac{1-a}{2}\sum_{k\geq 1}p_2(k)(p_3(k)+p_3(k-1))p_1(k-1) - \\
&- N_1 p_1(0)p_2(0)p_3(0), && (40)\\
\frac{H_{3d}}{\partial c_2} &= N_2 \sum_{k\geq 0}p_1(k)p_2(k)p_3(k) + N_2\frac{1-a}{2}\sum_{k\geq 1}(p_2(k-1)-p_2(k))(p_3(k)+p_3(k+1))\sum_{j=0}^{k-1}p_1(k), && (41)\\
\frac{H_{3d}}{\partial c_3} &= N_3 \sum_{k\geq 1}p_1(k)(p_3(k-1)-p_3(k))\sum_{j=0}^{k}p_2(j) - N_3\frac{1-a}{2}\sum_{k\geq 2}p_2(k)(p_3(k-2)-p_3(k))\sum_{j=0}^{k-1}p_1(k) - \\
&- N_3 p_1(0)p_2(0)p_3(0) - N_3\frac{1-a}{2}p_1(0)p_2(1)p_3(1)-1. && (42)
\end{aligned}$$

As $N\to\infty$, we have $\frac{c_2}{c_1}\to\gamma_2\in(0,1)$, $\frac{c_3}{c_1}\to\gamma_3\in(0,1)$. It follows that all of the above derivatives converge to 0 as $n\to\infty$, except $\frac{\partial H_{id}}{\partial c_i}$, $i=1,2,3$, that converge to -1. Hence the inverse of the Jacobian matrix $\left[\frac{\partial H_{ij}}{\partial c_j}\right]_{ij}$ converges to the negative identity matrix.

We also have

$$\begin{aligned}
\frac{H_{1d}}{\partial a} &= 0, && (43)\\
\frac{H_{2d}}{\partial a} &= -\frac{1}{2}\sum_{k\geq 1}p_3(k)(p_2(k)+p_2(k-1))\sum_{j=0}^{k-1}p_1(j) < 0, && (44)\\
\frac{H_{3d}}{\partial a} &= -\frac{1}{2}\sum_{k\geq 1}p_2(k)(p_3(k)+p_3(k-1))\sum_{j=0}^{k-1}p_1(j) < 0. && (45)
\end{aligned}$$

By the implicit function theorem we have

$$\lim_{N\to\infty} \frac{\frac{\partial c_2}{\partial a}}{\frac{H_{2d}}{\partial a}} = 1 \tag{46}$$

$$\lim_{N\to\infty} \frac{\frac{\partial c_3}{\partial a}}{\frac{H_{3d}}{\partial a}} = 1, \tag{47}$$

so for sufficiently large N we have $\frac{\partial c_2}{\partial a} < 0$ and $\frac{\partial c_3}{\partial a} < 0$, while $\frac{\partial c_1}{\partial a} = 0$. **Q.E.D**.

Proof of Proposition 3

R_c can be written as

$$R_c = \frac{Nb_1 F(c_1^*)}{N\sum_j b_j F(a_{j2}c_2^*) + Nb_1 F(c_1^*)}$$

Since, as $N \to \infty$, c_1^* and c_2^* converge to 0, we can apply Taylor's Theorem and write R_c as

$$R_c = \frac{b_1 c_1^*}{\sum_j b_j a_{j2} c_2^* + b_1 c_1^*},$$

then by substituting the definition $\tilde{b}$ and the limiting ratio stated in Proposition 1 we get

$$R_c = \frac{b_1\left(\frac{\tilde{b}}{b_1}\right)^{\frac{1}{3}}}{b_1\left(\frac{\tilde{b}}{b_1}\right)^{\frac{1}{3}} + \tilde{b}} = \frac{b_1^{\frac{2}{3}}}{b_1^{\frac{2}{3}} + \tilde{b}^{\frac{2}{3}}}.$$

From the proof of Proposition 2, an equilibrium with $L = 3 = n$ exists only if $b_3\gamma_{31} = b_2\gamma_{21} = b_1$. Then, R_d in such an equilibrium becomes

$$R_d = \frac{Nb_1c_1^*}{Nb_1c_1^* + Nb_2c_2^* + Nb_3c_3^*} = \frac{b_1}{b_1 + b_2\gamma_{21} + b_3\gamma_{31}} = \frac{1}{3}.$$

Q.E.D.

Proof of Corollary 1 When the equilibrium with $L = 3 = n$ exists, $\tilde{b}$ can be written as

$$\tilde{b} = b_2 + b_3 a_{32} = b_2\left(1 + a_{32}\frac{3 - a_{23}}{3 - a_{32}}\right).$$

Then $R_c > 1/2$ becomes

$$\frac{b_1}{\tilde{b}} > 1 \Leftrightarrow \frac{b_1}{b_2}\frac{3 - a_{23}}{3 + 2a_{32} - a_{23}a_{32}} > 1$$

The result follows from equating $a_{23} = a_{32} = a$, and $b_2 = b_3 = \frac{1-b}{2}$. **Q.E.D.**

Proof of Lemma 1.

Since finite sum of independent Poisson variables is also a Poisson variable, we can write $H_1(c)$ as follows

$$H_1(c) = \sum_{k_1\geq 0}\sum_{k_{-1}\geq 0} p_1(k_1)p_{-1}(k_{-1})\left(\frac{k_1+1}{k_1+k_{-1}+1} - \frac{k_1}{k_1+k_{-1}}\right) - c,$$

where $p_{-1}(k_{-1}) = \frac{e^{\tilde{r}_{-1}}\tilde{r}_{-1}^{k_{-1}}}{k_{-1}!}$, and $\tilde{r}_{-1} = \sum_{j=2}^{n} r_j$. Then, we can use a similar argument on the proof Lemma 1 by Herrera, Morelli and Palfrey (2014) to prove the result.

Now, fix $i \in \{2, \dots, n\}$ and $j \in \{1, \dots, L\}$. Then, we have

$$\sum_{k_1=0}^{\infty} \frac{(r_1 N)^{k_1}}{k_1!} \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'}}{\sum_{j=1}^{n} k_j}$$

$$= \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'}}{(r_1 N)^{\sum_{j=2}^{n} k_j}} \sum_{k_1=0}^{\infty} \frac{(r_1 N)^{\sum_{j=1}^{n} k_j}}{(\sum_{j=1}^{n} k_j) k_1!} = \begin{cases} \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'}}{(r_1 N)^{\sum_{j=2}^{n} k_j}} \int_0^{r_1 N} \rho^{\sum_{j=2}^{n} k_j - 1} e^{\rho} d\rho, & \sum_{j=2}^{n} k_j > 0 \\ \frac{\sum_{l=2}^{L} a_{il} |I_l|}{n}, & \sum_{j=2}^{n} k_j = 0. \end{cases}$$

and

$$\sum_{k_1=0}^{\infty} \frac{(r_1 N)^{k_1}}{k_1!} \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'} + a_{ij}}{\sum_{j=1}^{n} k_j + 1}$$

$$= \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'} + a_{ij}}{(r_1 N)^{\sum_{j=2}^{n} k_j + 1}} \sum_{k_1=0}^{\infty} \frac{(r_1 N)^{\sum_{j=1}^{n} k_j + 1}}{(\sum_{j=2}^{n} k_j + 1) k_1!} = \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'} + a_{ij}}{(r_1 N)^{\sum_{j=1}^{n} k_j + 1}} \int_0^{r_1 N} \rho^{\sum_{j=2}^{n} k_j} e^{\rho} d\rho.$$

This gives us

$$\begin{aligned} H_{ij}(c) &= e^{-N \sum_{j=1}^{n} r_j} \Bigg[\sum_{k_2 \geq 0, \dots, k_n \geq 0} \frac{\prod_{j=2}^{n} (r_i N)^{k_j}}{\prod_{j=2}^{n} k_j!} \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'} + a_{ij}}{(r_1 N)^{\sum_{j=2}^{n} k_j + 1}} \int_0^{r_1 N} \rho^{\sum_{j=1}^{n} k_j} e^{\rho} d\rho - \\ &- \sum_{\substack{k_2 \geq 0, \dots, k_n \geq 0 \\ k_2 + \dots + k_n > 0}} \frac{\prod_{j=2}^{n} (r_j N)^{k_j}}{\prod_{j=2}^{n} k_j!} \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'}}{(r_1 N)^{\sum_{j=2}^{n} k_j}} \int_0^{r_1 N} \rho^{\sum_{j=2}^{n} k_j - 1} e^{\rho} d\rho - \frac{\sum_{l=2}^{L} a_{il} |I_l|}{n} \Bigg] - c. \end{aligned} \tag{48}$$

Now,

$$\begin{aligned} &\sum_{k_2 \geq 0, \dots, k_n \geq 0} \frac{\prod_{i=2}^{n} (r_j N)^{k_j}}{\prod_{j=2}^{n} k_j!} \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'} + a_{ij}}{(r_1 N)^{\sum_{j=2}^{n} k_j + 1}} \rho^{\sum_{j=2}^{n} k_j} = \\ &\frac{1}{r_1 N} \sum_{k_2=0}^{\infty} \frac{\left(\frac{\rho r_2}{r_1}\right)^{k_2}}{k_2!} \dots \sum_{k_n=0}^{\infty} \frac{\left(\frac{\rho r_n}{r_1}\right)^{k_n}}{k_n!} \left(\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'} + a_{ij} \right) = \frac{e^{\frac{\rho}{r_1} \sum_{l=2}^{n} r_l}}{r_1 N} \left(\rho \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} r_{l'}}{r_1} + a_{ij} \right) \end{aligned} \tag{49}$$

and

$$\begin{aligned} &\sum_{\substack{k_2 \geq 0, \dots, k_n \geq 0 \\ k_2 + \dots + k_n > 0}} \frac{\prod_{j=2}^{n} (r_j N)^{k_j}}{\prod_{j=2}^{n} k_j!} \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} k_{l'}}{(r_1 N)^{\sum_{l=2}^{n} k_l}} \rho^{\sum_{j=2}^{n} k_j - 1} + \frac{\sum_{l=2}^{L} a_{il} |I_l|}{n} \\ &= \frac{1}{r_1} \sum_{k_2=0}^{\infty} \frac{\left(\frac{\rho r_2}{r_1}\right)^{k_2}}{k_2!} \dots \sum_{k_n=0}^{\infty} \frac{\left(\frac{\rho r_n}{r_1}\right)^{k_n}}{k_n!} \left(\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} r_{l'} \right) + \frac{\sum_{l=2}^{L} a_{il} |I_l|}{n} \\ &= \frac{e^{\frac{\rho}{r_1} \sum_{l=2}^{n} r_l}}{r_1} \left(\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} r_{l'} \right) + \frac{\sum_{l=2}^{L} a_{il} |I_l|}{n} \end{aligned} \tag{50}$$

It follows that

$$\begin{aligned} H_{ij}(c) = e^{-N \sum_{j=1}^{n} b_j F(c_j)} &\left[\left(\frac{a_{ij}}{r_1 N} - \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} r_{l'}}{r_1} \right) \int_0^{r_1 N} e^{\frac{\rho}{r_1} \sum_{l=1}^{n} r_l} d\rho + \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} r_{l'}}{r_1^2 N} \int_0^{r_1 N} \rho e^{\frac{\rho}{r_1} \sum_{l=1}^{n} r_l} d\rho \right] \\ &- e^{-N \sum_{j=1}^{n} b_j F(c_j)} \frac{\sum_{l=2}^{L} a_{il} |I_l|}{n} - c \\ = e^{-N \sum_{j=1}^{n} b_j F(c_j)} &\left(\frac{N \sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} r_{l'} - a_{ij}}{N (\sum_{j=1}^{n} b_j F(c_j))^2} + \frac{\sum_{l=2}^{L} a_{il} \sum_{l' \in I_l} r_{l'}}{N (\sum_{l=1}^{n} r_l)^2} - \frac{\sum_{l=2}^{L} a_{il} |I_l|}{n} \right) \\ &+ \frac{\sum_{l=1}^{L} (a_{ij} - a_{il}) \sum_{l' \in I_l} r'_l}{N (\sum_{j=1}^{n} b_j F(c_j))^2} - c. \end{aligned}$$

Q.E.D.

Proof of Proposition 4. The asymptotic equilibrium conditions are written as

$$\frac{\sum_{l=2}^{n} b_l c_l}{f(0)N(\sum_{l=1}^{n} b_l c_l)^2} - c_1 = 0, \qquad \frac{a_{i2} b_1 c_1}{f(0)N(\sum_{i=1}^{n} b_i c_i)^2} - c_i = 0, \quad i = 2, \dots n. \tag{51}$$

It follows that $\frac{c_j}{c_2} = \frac{a_{2j}}{a_{22}}$ for $j = 3, \dots, n$. Put $B = \sum_{j=2}^{n} b_j a_{j2}$. Conditions (51) become

$$\frac{Bc_2}{f(0)Na_{22}(b_1 c_1 + \frac{B}{a_{22}} c_2)^2} - c_1 = 0, \qquad \frac{a_{22} b_1 c_1}{f(0)N(b_1 c_1 + \frac{B}{a_{22}} c_2)^2} - c_2 = 0. \tag{52}$$

From this we obtain $c_1 = \frac{c_2}{a_{22}} \sqrt{\frac{B}{b_1}}$ and (13). **Q.E.D.**

Proof of Theorem 3. The expected number of votes for options 1 and 2 are given by

$$V_1^* = \frac{\sqrt{f(0)N} b_1^{\frac{3}{4}} \tilde{b}^{\frac{1}{4}}}{(\sqrt{b_1} + \sqrt{\tilde{b}})}, \qquad V_2^* = \frac{\sqrt{f(0)N} b_1^{\frac{1}{4}} \tilde{b}^{\frac{3}{4}}}{(\sqrt{b_1} + \sqrt{\tilde{b}})}. \tag{53}$$

We have

$$\frac{\partial V_1^*}{\partial \tilde{b}} = \frac{\sqrt{f(0)N} b_1^{\frac{3}{4}} \tilde{b}^{-\frac{3}{4}} (\sqrt{b_1} - \sqrt{\tilde{b}})}{4(\sqrt{b_1} + \sqrt{\tilde{b}})^2}, \qquad \frac{\partial V_2^*}{\partial \tilde{b}} = \frac{\sqrt{f(0)N} b_1^{\frac{1}{4}} (\tilde{b}^{\frac{1}{4}} + 3\tilde{b}^{-\frac{1}{4}} b_1^{\frac{1}{2}})}{4(\sqrt{b_1} + \sqrt{\tilde{b}})^2}.$$

The first value is positive (negative) if $b_1 > (<)\, \tilde{b}$, the second value is always positive. **Q.E.D.**

Proof of Proposition 5. The conditions on c_i for $i \neq 1$ will be symmetric, so we let $c_i = c_2$ for $i \neq 1, 2$. The equilibrium cutoff costs are given by

$$\frac{(1-b)c_2}{f(0)N(bc_1 + (1-b)c_2)^2} - c_1 = 0, \qquad \frac{c_1 b + \frac{(1-b)(1-a)(n-2)}{n-1} c_2}{f(0)N(bc_1 + (1-b)c_2)^2} - c_2 = 0. \tag{54}$$

This yields

$$c_2^2(1-b) - \frac{(1-b)(1-a)(n-2)}{n-1} c_1 c_2 - c_1^2 b = 0$$

or

$$c_2 = c_1 A$$

where A is given by (15). Finally, by substituting $c_2 = c_1 A$ into (54) we get (14). **Q.E.D.**

Proof of Theorem 4. We have

$$\frac{\partial c_1}{\partial a} = \frac{\sqrt{1-b}}{\sqrt{f(0)N}} \frac{1}{(b + (1-b)A)^2} \left(\frac{b}{2\sqrt{A}} - \frac{1-b}{2}\sqrt{A} \right) \left(-\frac{1}{2} - \frac{(1-a)(n-2)^2}{2(n-1)^2 \left(2A - \frac{(1-a)(n-2)}{n-1} \right)} \right)$$

and

$$\frac{\partial c_2}{\partial a} = \frac{\sqrt{1-b}}{\sqrt{f(0)N}} \frac{1}{(b + (1-b)A)^2} \left(\frac{3}{2} b\sqrt{A} + \frac{1}{2}(1-b)A^{\frac{3}{2}} \right) \left(-\frac{1}{2} - \frac{(1-a)(n-2)^2}{2n^2 \left(2A - \frac{(1-a)(n-2)}{n-1} \right)} \right).$$

The second expression is always negative, so the opposition cutoff cost is decreasing in a for all $b \in (0, 1)$ and for all n. The second expression is negative if and only if $A < \frac{b}{1-b}$. Hence, the incumbent cutoff cost c_1 will be increasing in a if

$b < \frac{1}{2}$; will be decreasing in a if $b > \frac{2n-3}{3n-4}$; and, if $b \in [\frac{1}{2}, \frac{2n-3}{3n-4}]$, it will be increasing on $[0, \tilde{a}]$ and decreasing on $[\tilde{a}, 1]$, where $\tilde{a} = \frac{(n-1)(1-2b)}{(n-2)(1-b)} + 1$. **Q.E.D.**

Proof of Proposition 6. Assume proportional representation. From (13), it follows that number of votes for options 1 and the total number of votes are Poisson variables with means approximated by (53) and

$$V_1^* + V_2^* = \sqrt{f(0)N} b_1^{\frac{1}{4}} \tilde{b}^{\frac{1}{4}},$$

and standard deviations being on the order of $\sqrt[\frac{1}{4}]{N}$. Hence, each of these variables, normalized by its mean, converges in distribution to 1, and (16) follows. The statement for (17) is obtained in a similar fashion. **Q.E.D.**

Proof of Corollary 2. We know that R_c decreases in a and R_d increases in a. For $a = 1$, we have $R_d = R_u = \sqrt{\frac{b}{1-b}}$. **Q.E.D.**

References

Abramowitz, Alan I and Kyle L Saunders. 2008. "Is polarization a myth?" *The Journal of Politics* 70(2):542–555.

Abramowitz, Alan I and Walter J Stone. 2006. "The Bush effect: Polarization, turnout, and activism in the 2004 presidential election." *Presidential Studies Quarterly* 36(2):141–154.

Abramowitz, Milton and Irene A Stegun. 1948. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Vol. 55 US Government printing office.

Adams, James, Jay Dow and Samuel Merrill. 2006. "The political consequences of alienation-based and indifference-based voter abstention: Applications to presidential elections." *Political Behavior* 28(1):65–86.

Arzumanyan, Mariam and Mattias K Polborn. 2017. "Costly voting with multiple candidates under plurality rule." *Games and Economic Behavior* 106:38–50.

Borgers, Tilman. 2004. "Costly voting." *American Economic Review* 94(1):57–66.

Bouton, Laurent and Micael Castanheira. 2012. "One person, many votes: Divided majority and information aggregation." *Econometrica* 80(1):43–87.

Bouton, Laurent, Micael Castanheira and Aniol Llorente-Saguer. 2017. "Multicandidate elections: Aggregate uncertainty in the laboratory." *Games and economic behavior* 101:132–150.

Callander, Steven and Catherine H Wilson. 2007. "Turnout, polarization, and Duverger's law." *The Journal of Politics* 69(4):1047–1056.

DiMaggio, Paul, John Evans and Bethany Bryson. 1996. "Have American's social attitudes become more polarized?" *American journal of Sociology* 102(3):690–755.

Druckman, James N, Erik Peterson and Rune Slothuus. 2013. "How elite partisan polarization affects public opinion formation." *American Political Science Review* pp. 57–79.

Faravelli, Marco and Santiago Sanchez-Pages. 2015. "(Don't) make my vote count." *Journal of Theoretical Politics* 27(4):544–569.

Fey, Mark. 1997. "Stability and coordination in Duverger's law: A formal model of preelection polls and strategic voting." *American Political Science Review* 91(1):135–147.

Fiorina, Morris P and Samuel J Abrams. 2008. "Political polarization in the American public." *Annu. Rev. Polit. Sci.* 11:563–588.

Goeree, Jacob K and Jens Grosser. 2007. "Welfare reducing polls." *Economic Theory* 31(1):51–68.

Herrera, Helios, Aniol Llorente-Saguer and Joseph C McMurray. 2019. "Information aggregation and turnout in proportional representation: a laboratory experiment." *Journal of Public Economics* 179:104051.

Herrera, Helios, Massimo Morelli and Salvatore Nunnari. 2016. "Turnout across democracies." *American Journal of Political Science* 60(3):607–624.

Herrera, Helios, Massimo Morelli and Thomas Palfrey. 2014. "Turnout and power sharing." *The Economic Journal* 124(574):F131–F162.

Iyengar, Shanto and Sean J Westwood. 2015. "Fear and loathing across party lines: New evidence on group polarization." *American Journal of Political Science* 59(3):690–707.

Kartal, Melis. 2015. "A comparative welfare analysis of electoral systems with endogenous turnout." *The Economic Journal* 125(587):1369–1392.

Krasa, Stefan and Mattias K Polborn. 2009. "Is mandatory voting better than voluntary voting?" *Games and Economic Behavior* 66(1):275–291.

Krishna, Vijay and John Morgan. 2015. "Majority rule and utilitarian welfare." *American Economic Journal: Microeconomics* 7(4):339–75.

Mason, Lilliana. 2015. ""I disrespectfully agree": The differential effects of partisan sorting on social and issue polarization." *American Journal of Political Science* 59(1):128–145.

McCright, Aaron M and Riley E Dunlap. 2011. "The politicization of climate change and polarization in the American public's views of global warming, 2001–2010." *The Sociological Quarterly* 52(2):155–194.

Moral, Mert. 2017. "The bipolar voter: On the effects of actual and perceived party polarization on voter turnout in European multiparty democracies." *Political Behavior* 39(4):935–965.

Myerson, Roger B. 2000. "Large poisson games." *Journal of Economic Theory* 94(1):7–45.

Myerson, Roger B and Robert J Weber. 1993. "A theory of voting equilibria." *American Political Science Review* 87(1):102–114.

Ortoleva, Pietro and Erik Snowberg. 2015. "Overconfidence in political behavior." *American Economic Review* 105(2):504–35.

Palfrey, Thomas R. 1988. "A mathematical proof of Duverger's law.".

Palfrey, Thomas R and Howard Rosenthal. 1983. "A strategic calculus of voting." *Public choice* 41(1):7–53.

Palfrey, Thomas R and Howard Rosenthal. 1985. "Voter participation and strategic uncertainty." *American political science review* 79(1):62–78.

Riker, William H. 1982. "The two-party system and Duverger's law: An essay on the history of political science." *American political science review* 76(4):753–766.

Rogowski, Jon C. 2014. "Electoral choice, ideological conflict, and political participation." *American Journal of Political Science* 58(2):479–494.

Tyson, Scott A. 2016. "Information acquisition, strategic voting, and improving the quality of democratic choice." *The journal of politics* 78(4):1016–1031.

Vorobyev, Dmitriy. 2018. "Information Disclosure in Elections with Sequential Costly Participation.".

Xefteris, Dimitrios. 2019. "Strategic voting when participation is costly." *Games and Economic Behavior* 116:122–127.