НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

О выборе победителя в турнире: теория и приложения

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Alternatives, comparisons, choices

X – the *general set* of alternatives.

A – the *feasible set* of alternatives: $A \subseteq X \wedge A \neq \varnothing$. The feasible set is a variable.

R – results of binary comparisons, $R \subseteq X \times X$.

R is presumed to be complete: $\forall\, x \in X, \forall\, y \in X, (x, y) \in R \vee (y, x) \in R$.

$R|_A = R \cap A \times A$ – restriction of R onto A.

$(A, R|_A)$ – *abstract game*.

P – asymmetric part of R, $P \subseteq R$: $(x, y) \in P \Leftrightarrow ((x, y) \in R \wedge (y, x) \notin R)$.

If $P|_A$ is complete, $\forall\, x \in X, \forall\, y \in X \wedge y \neq x, (x, y) \in P \vee (y, x) \in P$, then

$(A, P|_A)$ – *tournament*.

Tournament solutions

A *tournament solution* S is a choice correspondence $S(A, P)$: $2^X\backslash\varnothing \times 2^{X\times X} \to 2^X$

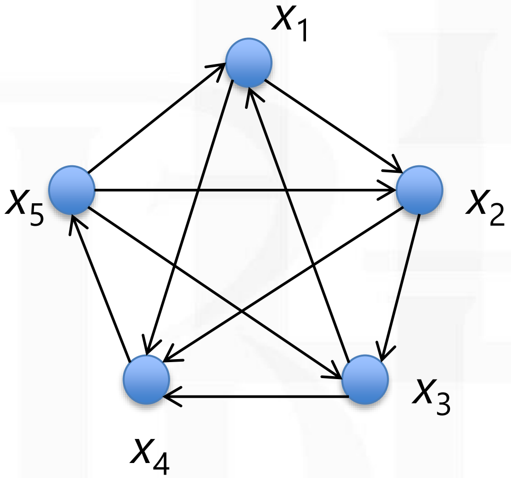
that has the following properties:

0. Locality: $S(A, P)=S(P|_A) \subseteq A$

1. *Nonemptiness*: $\forall A, \forall P, \; S(P|_A) \neq \varnothing$;
2. *Neutrality*: permutation of alternatives' names and choice commute;
3. *Condorcet consistency*: if there is the Condorcet winner w for $P|_A$ then $S(P|_A) = \{w\}$.

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	0	1	0
x_2	0	0	1	1	0
x_3	1	0	0	1	0
x_4	0	0	0	0	1
x_5	1	1	1	0	0

Tournament matrix



Tournament digraph

Properties a.k.a. Axioms

- *Idempotency*: $\forall A$, $S(S(A))=S(A)$.
- *The Aizerman-Aleskerov condition*: $\forall A$, $\forall B$, $S(A)\subseteq B\subseteq A \Rightarrow S(B)\subseteq S(A)$.
- *generalized Nash independence of irrelevant alternatives* (*ind. of outcasts*):
- $\forall A$, $\forall B$, $S(A)\subseteq B\subseteq A \Rightarrow S(B)=S(A)$.

NIIA $\Leftrightarrow$ *Idempotency* $\wedge$ *the Aizerman-Aleskerov condition*

- *Monotonicity (monotonicity w.r.t. results*):

$\forall P_1,P_2\subseteq X^2$, $\forall A\subseteq X$, $\forall x\in S(P_1|_A)$, $(P_1|_{A\setminus\{x\}}=P_2|_{A\setminus\{x\}}\wedge\forall y\in A$, $xP_1y \Rightarrow xP_2y) \Rightarrow x\in S(P_2|_A)$

- *Independence of irrelevant results (ind. of losers)*:

$\forall P_1,P_2\subseteq X^2$, $\forall A\subseteq X$, $(\forall x\in S(P_1|_A),\forall y\in A$, $((xP_1y \Leftrightarrow xP_2y) \wedge (yP_1x \Leftrightarrow yP_2x)) \Rightarrow S(P_1|_A)=S(P_2|_A)$

- *Computational simplicity*: There is a polynomial algorithm for computing S.

Solutions

Uncovered set $UC = \{x \in A \mid \forall\, y \in A,\, yPx \Rightarrow \exists\, z \in A: xPzPy\}$

Copeland set $C = \text{argmax}\, |\{y \in A \mid xPy\}|$

Slater set $SL = \{\max(L_k) \mid L_k \in \text{argmin}\, \kappa(L_k, P)\}$,

where $L_k \subseteq A \times A$ – a linear order, $\kappa(L_k, P)$ – the Kendall distance

Banks set $B = \{\max(L_k) \mid L_k \subseteq P \subseteq A \times A$ – maximal chain in $P\}$

Minimal covering set $MC,\ \forall\, x \in MC,\ x \in UC(P|_{MC}) \wedge \forall\, x \notin MC,\ x \notin UC(P|_{MC \cup \{x\}})$

Bipartisan set BP = support(*Nash Equilibrium*$(G(P|_A))$), where $G(P|_A)$ is

a two-player zero-sum non-cooperative game on a tournament $P|_A$

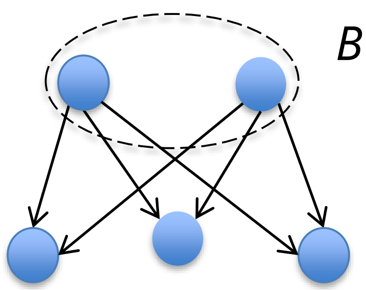
Stable sets

A nonempty subset *B* of *A* is called

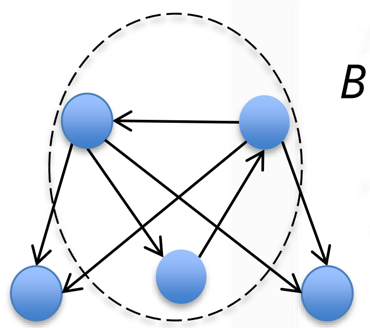
Dominant if $\forall\, x \in A \backslash B,\ \forall\, y \in B: yPx$

Dominating if $\forall\, x \in A,\ \exists\, y \in B: yPx$

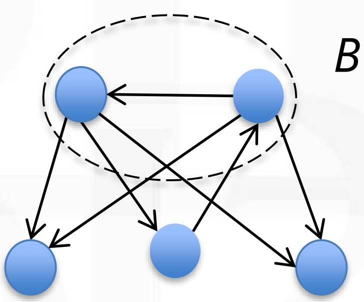
Externally stable if $\forall\, x \in A \backslash B,\ \exists\, y \in B: yPx$



P-dominant



P-dominating



P-ext. stable

Minimal stable sets

A set *B* is called *minimal* with respect to a given property if

B has the property and none *of* B's proper nonempty subsets does.

Tournament solutions: the union of all minimal

Dominant sets *TC* a.k.a. the *Top cycle*

Dominating sets *D*

Externally stable sets *ES*

Axiomatic analysis

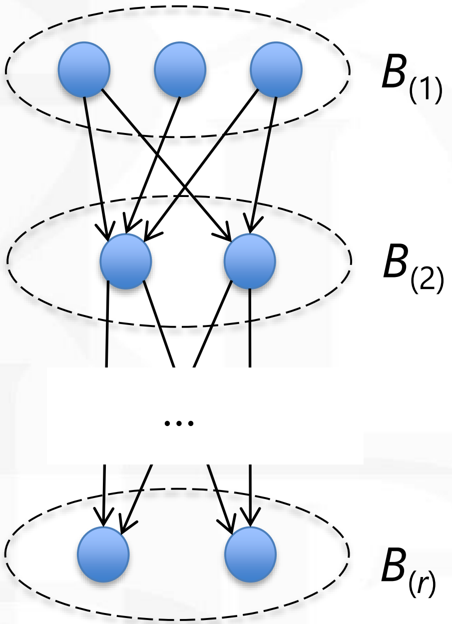
	UC	*C*	*SL*	*B*	*MC*	*BP*	*TC*	*D*	*ES*
Idempotence	**NO**	**NO**	**NO**	**NO**	**YES**	**YES**	**YES**	**NO**	**YES**
AA property	**YES**	**NO**	**NO**	**YES**	**YES**	**YES**	**YES**	**NO**	**YES**
Outcast (Nash independence)	**NO**	**NO**	**NO**	**NO**	**YES**	**YES**	**YES**	**NO**	**YES**
Monotonicity	**YES**	**YES**	**YES**	**YES**	**YES**	**YES**	**YES**	**NO**	**YES**
Independence of losers	**NO**	**NO**	**NO**	**NO**	**YES**	**YES**	**YES**	**NO**	**YES**
Computational simplicity	**YES**	**YES**	**NO**	**NO**	**YES**	**YES**	**YES**	**YES**	**YES**

Ranking based on a tournament solution

Suppose, we are interested in ranking alternatives from A.

Then we may use the following procedure:

- Tournament solution $S(P, A)$ choses the set $B_{(1)}$ of the best alternatives in A, $B_{(1)}=S(P, A)$.
- Exclude these alternatives from A and apply S to the rest. $B_{(2)}=S(P, A\backslash B_{(1)})=S(P, A\backslash S(P, A))$ will be the set of the second-best alternatives in A.
- By repeated exclusion of the best alternatives determined at each step of the procedure the set A is separated into groups $B_{(r)}=S(P, A\backslash(B_{(r-1)}\cup B_{(r-2)}\cup\ldots\cup B_{(2)}\cup B_{(1)}))$, and that is the ranking.
- Let $r = r(x, P)$ denote the rank of x in this ranking.



The properties of the ranking rule based on sorting either by *ES* or by *RES*

- ***Weak Pareto principle***: if x Pareto dominates y, then xQ (P)y.

- ***Weak monotonicity w.r.t the individual preferences*** Π_i (*Smith's monotonicity*):

$$(\mathbf{\Pi}|_{A\setminus\{x\}} = \mathbf{\Pi}'|_{A\setminus\{x\}} \wedge \forall i \in G, \forall y \in A, x\,\Pi_i y \Rightarrow x\,\Pi_i' y) \Rightarrow$$

$$\Rightarrow (\forall y \in A, xQ\,(P)y \Rightarrow xQ\,(P')y\,)$$

Independence of irrelevant classes of alternatives

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Спасибо
за внимание!

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