Weakening of the Nash equilibrium concept: existence and application to the Hotelling model

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Two-person game: basic notions

Consider 2-person non-cooperative game in the normal form

$$G = (i \in \{1,2\}; s_i \in S_i; u_i : S_1 \times S_2 \to R).$$

Definition (profitable deviation)

A profitable deviation of player i at strategy profile $s = (s_i, s_{-i})$ is a strategy s_i' such that

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Definition (NE)

A strategy profile s is a Nash Equilibrium if no player has a profitable deviation.

Treats and security, Iskakov & Iskakov, 2012

Definition (treat)

A threat of player i to player $-i$ at strategy profile s is a strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$ and $u_{-i}(s'_i, s_{-i}) < u_{-i}(s_i, s_{-i})$. The strategy profile s is said to pose a threat from player i to player $-i$.

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A profitable deviation s_i' of player i at s is secure if for any threat s'_{-i} of player $-i$ at profile (s_i', s_{-i}) $\quad u_i(s_i', s'_{-i}) \geq u_i(s_i, s_{-i})$.

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Definition (EinSS)

A strategy profile is an equilibrium in secure strategies if it is secure and no player has a profitable secure deviation.

Nash-2 equilibrium

Definition (alternative: secure deviation)

A profitable deviation s'_i of player i at s is secure if for any strategy s'_{-i} of player $-i$ such that $u_{-i}(s'_i, s'_{-i}) > u_{-i}(s'_i, s_{-i}) \quad u_i(s'_i, s'_{-i}) \geq u_i(s_i, s_{-i})$.

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NE-2 may be not secure.

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Proposition (I)

- *Any NE is an EinSS. (Iskakov & Iskakov, 2012)*
- *Any EinSS is a NE-2.*

Strictly competitive games

Definition (strictly competitive game)

A two-person game G is strictly competitive if for every two strategy profiles s and s'

$$u_i(s) \geq u_i(s') \quad \Longrightarrow \quad u_{-i}(s) \leq u_{-i}(s').$$

Examples: zero-sum games, constant-sum games...

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Denote the guaranteed gains of players 1 and 2 by

$$\underline{V_1} = \max_{s_1} \min_{s_2} u_1(s_1, s_2).$$

$$\underline{V_2} = \max_{s_2} \min_{s_1} u_2(s_1, s_2).$$

Necessary and sufficient conditions of the NE-2 existence for strictly competitive games

Theorem (necessary condition of NE-2)

If strategy profile s is a NE-2, then $u_i(s) \geq \underline{V_i}$ *for both players.*

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If a strategy profile s is such that for each player $u_i(s) > \underline{V_i}$*, then s is a NE-2.*

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Theorem (sufficient condition of NE-2)

If a strategy profile s is such that for each player $u_i(s) > \underline{V_i}$, then s is a NE-2.

Theorem (criterion of NE-2)

Assume a strategy profile $s^* = (s_i^*, s_{-i}^*)$ is such that $u_i(s^*) = \underline{V_i}$ for $i = 1$ or $i = 2$.
s^* is NE-2 if and only if for any $s_i \in \tilde{S}_i = \{s_i : \min_{s_{-i}} u_i(s_i, s_{-i}) = \underline{V_i}\}$

$$u_i(s_i, s_{-i}^*) = \underline{V_i}.$$

"Continuous" case

Definition (path-connected space)

The topological space X is said to be path-connected if for any two points $x, y \in X$ there exist a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Example: convex set in $\mathbb{R}^n$.

Theorem (NE-2 existence in continuous strictly comp. games)

Assume G is a two-person strictly competitive game. Strategy sets S_1 and S_2 are compact and path-connected. Payoff functions u_1 and u_2 are continuous.
Then there exist NE-2 in G (in pure strategies).

The Hotelling model on the unit circle

Location is an angle $\alpha \in [0; \pi]$ between two firms 1 and 2.

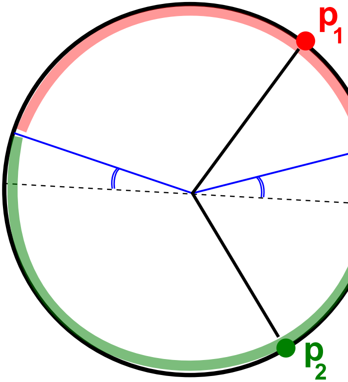

Fig.1

Price-setting game

$$v_1(p_1, p_2) = \begin{cases} p_1(\pi + p_2 - p_1), & \textit{if} \, |p_1 - p_2| \leq \alpha, \\ 2\pi p_1, & \textit{if } p_1 < p_2 - \alpha, \\ 0, & \textit{if } p_1 > p_2 + \alpha, \end{cases}$$

$$v_2(p_1, p_2) = \begin{cases} p_2(\pi + p_1 - p_2), & \textit{if} \, |p_1 - p_2| \leq \alpha, \\ 2\pi p_2, & \textit{if } p_2 < p_1 - \alpha, \\ 0, & \textit{if } p_2 > p_1 + \alpha. \end{cases}$$

Assume $\bar{p}_2$ is fixed

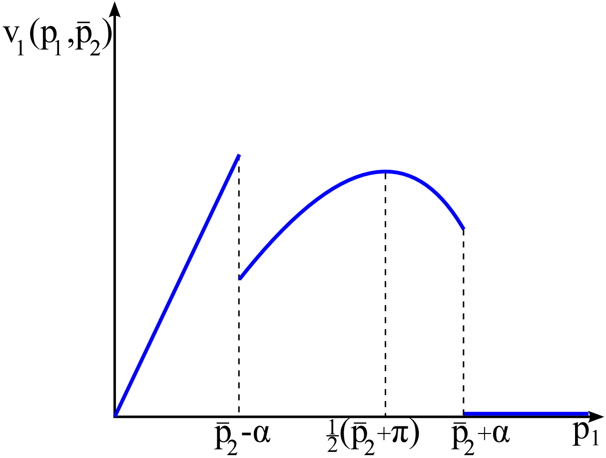

Fig.2

NE and EinSS in the Hotelling game

Theorem (NE, Hotelling)

For $\alpha \in [\frac{\pi}{2}, \pi]$ *the unique NE is* $p_1^* = p_2^* = \pi$. $v_1 = v_2 = \pi^2$.

For $\alpha = 0$ *the unique NE is* $p_1^* = p_2^* = 0$. $v_1 = v_2 = 0$.

For $\alpha \in (0, \frac{\pi}{2})$ *NE does not exist.*

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Theorem (EinSS, Hotelling)

For $\alpha \in [\frac{\pi}{2}; \pi]$ *the unique EinSS is* $p_1^* = p_2^* = \pi$. $v_1 = v_2 = \pi^2$.

For $\alpha \in [0; \frac{\pi}{2})$ *the unique EinSS is* $p_1^* = p_2^* = 2\alpha$. $v_1 = v_2 = 2\pi\alpha < \pi^2$.

NE-2 in the Hotelling game

Theorem (NE-2, Hotelling)

For all locations $\alpha \in [0;\pi]$ *the profile* (p_1, p_2) *is NE-2 if*

$$|p_1 - p_2| \le \alpha,$$

$$p_1 \ge \frac{p_2 + \pi}{2} \text{ and } p_2 \ge \frac{p_1 + \pi}{2}.$$

For $\alpha \in [0; \frac{\pi}{2})$ *profile* $(2\alpha, 2\alpha)$ *is also an isolated NE-2.*

NE-2 prices

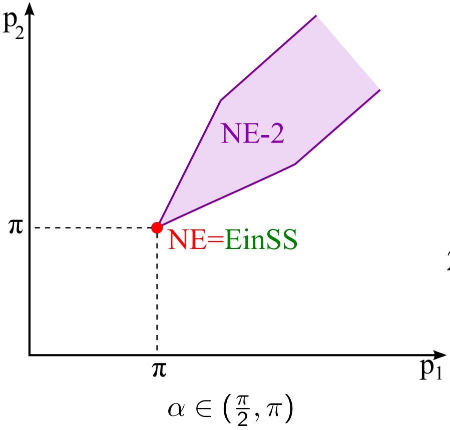
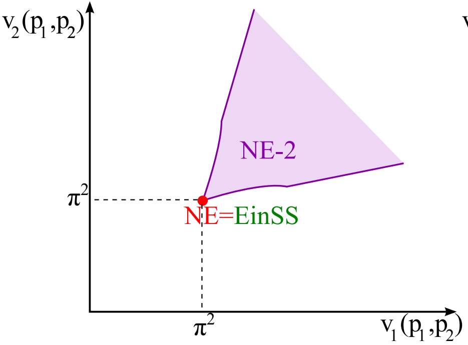


Fig.3

NE-2 gains



$\alpha \in (\frac{\pi}{2}, \pi)$ $\alpha \in (0, \frac{\pi}{2})$

Fig.4

Related references

1. d'Aspremont C., Gabszewicz J., Thisse J.-F. *On Hotelling's "Stability in Competition"* // Econometrica. 1979. Vol. 47. No. 5. P. 1145-1150.
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Related references

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Thank you for your attention!