Soliton in an extended nonlinear Schrödinger equation with stimulated scattering on damping low-frequency waves and nonlinear dispersion

E M Gromov

National Research University Higher School of Economics,
25/12 Bolshaja Pecherskaja Ulitsa, Nizhny Novgorod, 603155, Russia

E-mail: egromov@hse.ru

Abstract. Dynamics of solitons is considered in an extended nonlinear Schrödinger equation, including a pseudo-stimulated-Raman-scattering (pseudo-SRS) term (scattering on damping low-frequency waves, nonlinear dispersion and inhomogeneity of the spatial second-order dispersion (SOD). It is shown that wave-number downshift by the pseudo-SRS may be compensated by upshift provided by spatially increasing SOD with taking into account nonlinear dispersion. The equilibrium state is stable for negative parameter of nonlinear dispersion and unstable for positive one. The analytical solutions are verified by comparison with numerical results

1. Introduction

The great interest to the dynamics of solitons is motivated by their ability to travel long distances keeping the shape and transferring the energy and information. Soliton solutions are relevant to nonlinear models in various areas of physics which deal with the propagation of intensive wave fields in dispersive media: optical pulses and beams in fibers and spatial waveguides, electromagnetic waves in plasma, surface waves on deep water, etc. [1-7].

Dynamics of long high-frequency (HF) wave packets is described by the second-order nonlinear dispersive wave theory. The fundamental equation of the theory is the nonlinear Schrödinger equation (NLSE) [8,9], which includes the second-order dispersion (SOD) and cubic nonlinearity (self-phase modulation). Soliton solutions in this case arise as a result of the balance between the dispersive stretch and nonlinear compression of wave packets.

To solve many applied problems there is the necessary decreasing of solitons`s space size. Such decreasing is accompanied, as usually, by stimulated scattering on low-frequency (LF) media perturbations. To this time stimulated scattering on spatially homogeneous LF time modes (stimulated Raman scattering (SRS)) was considered in details [1]. SRS is described in extended NLSE by term with time delay of nonlinear kerr response. For localized nonlinear wave packets (solitons), the SRS gives rise to the downshift of the soliton frequency [10] and eventually to destabilization of the solitons. The compensation of the SRS was studied to this time distally [10-21].

For a series media the propagation of short solitons is accompanied by arising of damping LF waves. These LF modes are internal waves in the stratified fluid and ion-sound waves in the plasma.

Model for describing of stimulated scattering of HF waves on damping LF waves, named as pseudo-stimulated-Raman-scattering (pseudo-SRS), was proposed in [22-24]. Taking into account the wave factor of stimulated LF perturbations significantly varieties the dynamics of short HF solitons in these media. The pseudo-SRS leads to the self-wavenumber downshift, similar to what is well known in the temporal domain [1,10-21] and, eventually, to destabilization of the solitons. The model equation elaborated in [22-24] also included smooth spatial variation of the SOD, accounted for by a spatially decreasing SOD coefficient, which leads to an increase of the soliton's wave-number, making it possible to compensate the effect of the pseudo-SRS on the soliton by the spatially inhomogeneous SOD. The equilibrium between the pseudo-SRS and decreasing SOD gives rise to stabilization of the soliton's wave-number spectrum. However, the consideration was carried out in disregard of the nonlinear dispersion.

In this work the soliton dynamics is considered in the frame of extended NLSE with a pseudo-SRS, decreasing dispersion and with taking into account nonlinear dispersion. Shows that equilibrium state between the pseudo-SRS and decreasing SOD is stable focus for negative nonlinear dispersion, and unstable focus for positive nonlinear dispersion.

2. The basic equation and integral relations

The great Let's consider the dynamics of the HF wave field $U(\xi,t)\exp(i\omega t - i\kappa\xi)$ in the frame of extension NSE with pseudo-SRS, nonlinear dispersion and inhomogeneous SOD:

$$2i\frac{\partial U}{\partial t}+\frac{\partial}{\partial \xi}\left[q(\xi)\frac{\partial U}{\partial \xi}\right]+2U|U|^2+2i\beta\frac{\partial\left(U|U|^2\right)}{\partial \xi}+\mu U\frac{\partial\left(|U|^2\right)}{\partial \xi}=0\,, \tag{1}$$

where $q(\xi)$ is the SOD, μ is the pseudo-SRS, β is the nonlinear dispersion (self-stepping). Equation (1) with zero conditions on infinity $U\big|_{\xi\to\pm\infty}\to 0$ has the following integrals:

$$\frac{\mathrm{d}N}{\mathrm{d}t}\equiv\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}|U|^2 d\xi=0\,, \tag{2}$$

$$2\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}K|U|^2 d\xi=-\mu\int_{-\infty}^{\infty}\left[\frac{\partial\left(|U|^2\right)}{\partial \xi}\right]^2 d\xi-\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\mathrm{d}\xi}\left|\frac{\partial U}{\partial \xi}\right|^2 d\xi\,, \tag{3}$$

$$N\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}t}\equiv\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\xi|U|^2 d\xi=\int_{-\infty}^{+\infty}qK|U|^2 d\xi+\frac{3}{2}\beta\int_{-\infty}^{+\infty}|U|^4 d\xi\,, \tag{4}$$

where $U\equiv|U|\exp(i\phi)$, $K\equiv\partial\phi/\partial\xi$ is the local wave-number of wave packet.

3. Analytical results

For analysis of the system (2)-(4) we can found from imaginary part of Eq.(1)

$$\frac{\partial\left(|U|^2\right)}{\partial t}+\frac{\partial}{\partial \xi}\left(qK|U|^2+\frac{3}{2}\beta|U|^4\right)=0\,. \tag{5}$$

Let present wave-number in (5) as $K(\xi,t)=-(3/2)\beta|U(\xi,t)|^2/q(\xi)+\tilde{K}(\xi,t)$, where $\tilde{K}$ is described by equation

$$\frac{\partial\left(|U|^2\right)}{\partial t}+\frac{\partial\left(q\tilde{K}|U|^2\right)}{\partial \xi}=0\,. \tag{6}$$

To found $\tilde{K}$ we assume that the scales of the inhomogeneity of the SOD term and additional local wave-number $\tilde{K}$ are much larger than the size of the wave-packet envelope, hence the spatial variation of the $\tilde{K}$ may be locally approximated by the linear function of the coordinate, $\tilde{K}(\xi,t) \approx \tilde{K}(\bar{\xi},t) + \left(\partial\tilde{K}/\partial\xi\right)_{\bar{\xi}}\left(\xi-\bar{\xi}\right)$. Then we obtain from Eq. (6) under condition $\left(\partial|U|/\partial\xi\right)_{\bar{\xi}} = 0$ (which means that the peak of the soliton's amplitude is located at its center):

$$\left(\frac{\partial\tilde{K}}{\partial\xi}\right)_{\bar{\xi}} = -\left(\frac{2}{q|U|}\frac{\partial|U|}{\partial t} + \frac{1}{q}\frac{\mathrm{d}q}{\mathrm{d}\xi}\tilde{K}\right)_{\bar{\xi}}. \tag{7}$$

Further, replacing $\tilde{K}(\xi,t)$ for soliton-like wave packets by $\tilde{K}(\xi,t) \equiv k(t)$, we have for wave-number distribution

$$K(\xi,t) = -\frac{3}{2}\frac{\beta}{q(\xi)}|U(\xi,t)|^2 + k(t). \tag{8}$$

Spatial variation of the wave-number is caused by the nonlinear dispersion β. With taking into account (2) and (8), and neglecting terms order β^2, system (2)-(4) is reduced to:

$$2N\frac{\mathrm{d}k}{\mathrm{d}t} = -\frac{\mu L_0}{n^3} - \frac{q'Z_0}{n^2} - q'Nk^2 - \frac{3q'\beta M_0}{q_0}\frac{k}{n^2}, \quad \frac{\mathrm{d}n}{\mathrm{d}t} = q'nk, \tag{9}$$

where $q'(\bar{\xi}) = \left(dq/d\xi\right)_{\bar{\xi}}$, $n = q(\bar{\xi})/q_0$, $q_0 = q(0)$; $L_0 = \int_{-\infty}^{\infty}\left[\partial\left(|U(\xi,t=0)|^2\right)/\partial\xi\right]^2 d\xi$, $Z_0 = \int_{-\infty}^{\infty}\left[\partial|U(\xi,t=0)|/\partial\xi\right]^2 d\xi$, $M_0 = \int_{-\infty}^{+\infty}|U(\xi,t=0)|^4 d\xi$ are initial integral moments of wave packet. Equilibrium state of Eqs.(9) is

$$k_* = 0, n_* = -\mu L_0/q'Z_0. \tag{10}$$

In particular for

$$\mu = \mu_* \equiv -q'Z_0/L_0 \tag{11}$$

equilibrium state (10) achieved for $n_* = n_0 \equiv 1$, coinciding with initial wave-packet parameter. In the equilibrium regime, the wave packet U propagates with the integral moments, N, L_0, Z_0 and wave-number K, keeping their initial values, N, L_0, Z_0, $K(\xi,t=0) = -(3/2)\beta|U(\xi,t=0)|^2/q_0$. For linearly decreasing SOD, $q' = \text{const} < 0$, after replacements, $y \equiv k\sqrt{N/Z_0}$, $\theta \equiv -tq'\sqrt{Z_0/N}$, system (9) is reduced to

$$2\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{\lambda}{n^3} + \frac{1}{n^2} + y^2 + \delta\frac{y}{n^2}, \quad \frac{\mathrm{d}n}{\mathrm{d}\theta} = -ny, \tag{12}$$

where $\lambda \equiv \mu L_0/(-q'Z_0)$, $\delta \equiv 3\beta M_0/\left(q_0\sqrt{Z_0 N}\right)$. For $\delta < 0$ the equilibrium state of Eqs.(12) is stable focus, $\delta = 0$: center, $\delta > 0$: unstable focus. The trajectories following from Eqs. (12) on plane (n, y) are shown in Figure 1 for initial conditions $y_0 = 0$, $n_0 = 1$ with different δ and λ. For $\lambda = 1$ trajectories coincide with the initial point $(0;1)$.

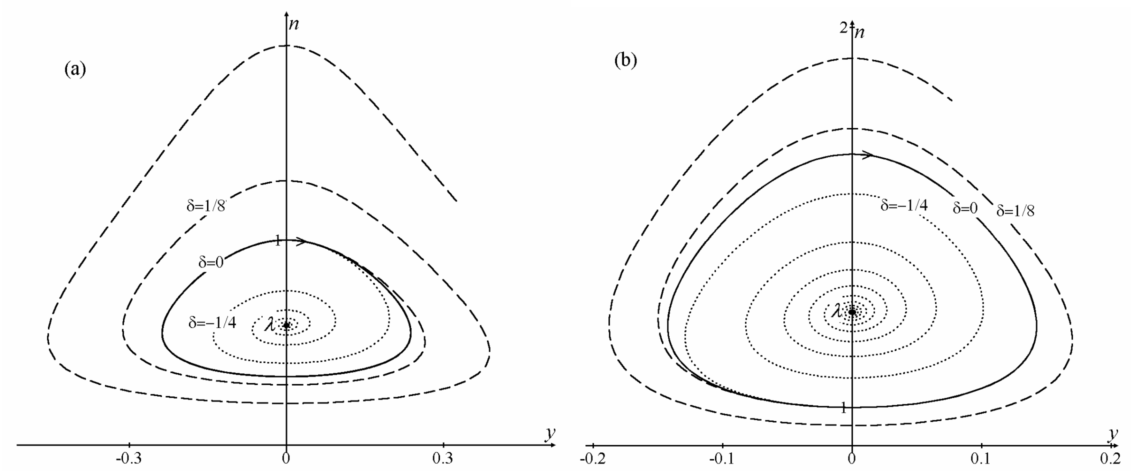

Figure 1. The trajectories following from (12) on plane (y,n) for initial conditions $y_0 = 0$, $n_0 = 1$ with different λ [(a): $\lambda = 3/4$, (b): $\lambda = 5/4$] and different δ.

4. Numerical results

We simulated solutions of the initial-value problem for the wave packet, $U(\xi, t = 0) = \exp[-i(3/2)\beta \tanh\xi]\operatorname{sech}\xi$ (with initial distribution of wave-number $K(\xi, t = 0) = -(3/2)\beta \operatorname{sech}^2\xi \equiv -(3/2)\beta |U(\xi, t = 0)|^2$) in the framework of Eq. (1) for $q(\xi) = 1 - \xi/10$, and different values of μ, β. The analytically predicted equilibrium value of the spatial-SRS coefficient (11) for this initial pulse is $\mu_* = 1/16$. In direct simulations, the initial pulse for $\mu = 1/16$ and different β is a stationary localized distribution (figure 2).

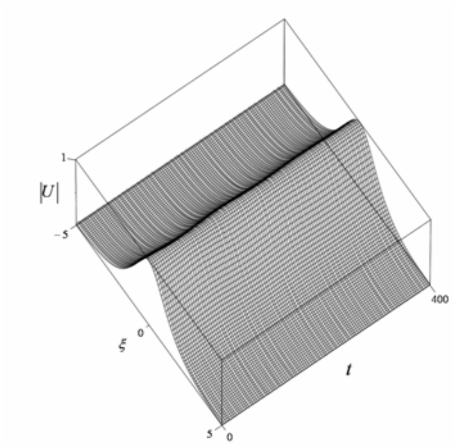

Figure 2. The numerically simulation evolution of modulus of the wave-packet envelope versus ξ, t for $\mu = 1/16 \equiv \mu_*$ with different values of β.

At values of the pseudo-SRS coefficient different from μ_*, given by Eq. (11), the simulations produce nonstationary solitons, see an example for $\mu = (3/4)\mu_* \equiv 3/64$ in figures 3 and 4.

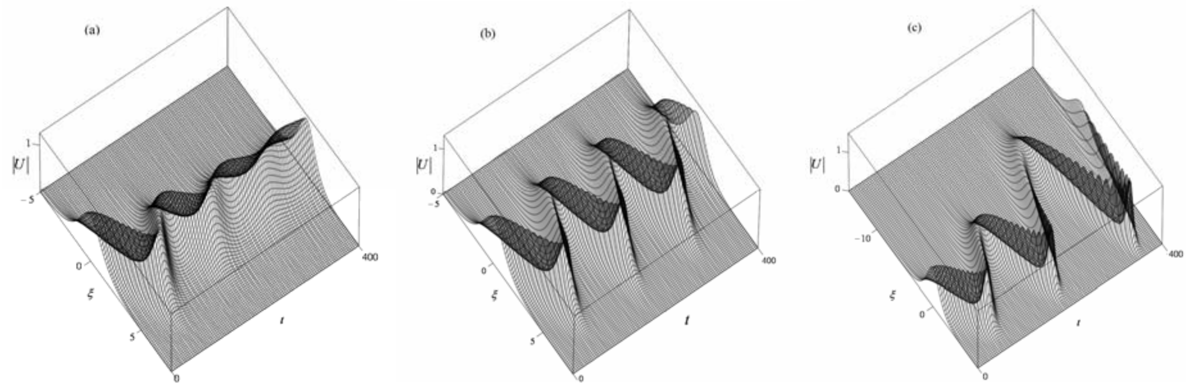

Figure 3. The numerically simulation evolution of modulus of the wave-packet envelope versus ξ, t for $\mu = (3/4)\mu_* \equiv 3/64$ with different values of β [(a): $\beta = -1/16$, (b): $\beta = 0$, (c): $\beta = 1/32$].

In figure 4, numerical results produced, as functions of time, by the simulations for the value of point coordinate of the maximum modulus of the wave-packet's shape $\xi_{\rm m}$ ($\max|U(\xi,t)| = |U(\xi_{\rm m},t)|$), are compared with the analytical counterparts of the mass-center wave-packet envelope $\overline{\xi} \equiv q_0(n-1)/q'$ obtained from Eqs. (9) for $\mu = (3/4)\mu_* \equiv 3/64$ and different values of β.

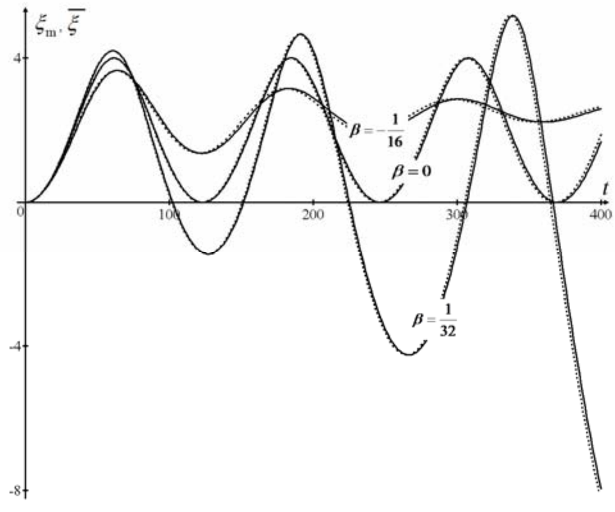

Figure 4. Numerical results (solid curves) for the value of point coordinate of the maximum modulus of the wave-packet's shape $\xi_{\rm m}$ and analytical results (dashed curves) for the mass-center wave-packet envelope $\overline{\xi}$ versus time for $\mu = (3/4)\mu_* \equiv 3/64$, and different values of β.

In figure 5, numerical results produced, as functions of time, by the simulations for the value $k_{\rm num}(t) = K(\xi_{\rm m},t) + 3\beta|U(\xi_{\rm m},t)|^2/(2q(\xi_{\rm m}))$ at the point of maximum modulus wave-packet's shape $\xi_{\rm m}$, are compared with the analytical counterparts of wave-number $k(t)$ (see relation (8)) obtained from Eqs. (9) for $\mu = (3/4)\mu_* \equiv 3/64$ and different values of β. Close agreement between the analytical and numerical results is demonstrated by the figure. A similar picture is observed at other values of the parameters.

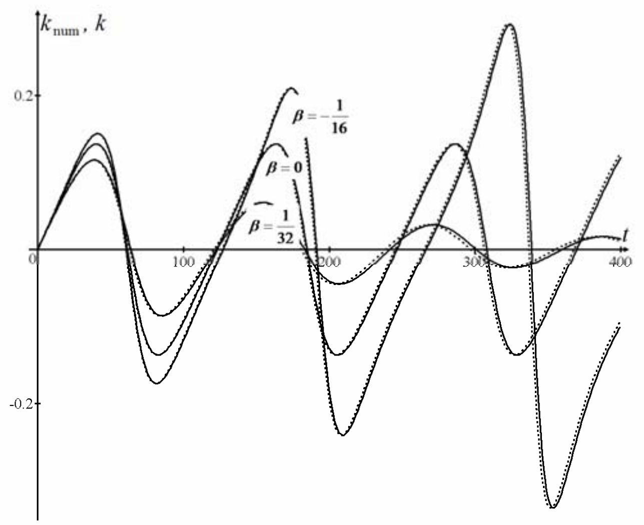

Figure 5. Numerical results (solid curves) for the value k_{num} at the point of maximum modulus of the wave-packet's shape ξ_m and analytical results (dashed curves) for the value k versus time obtained from (9) for $\mu = (3/4)\mu_* \equiv 3/64$, and different values of β.

5. Conclusion

In this work, we studied the soliton dynamics in the framework of the extended inhomogeneous NLSE, includes the pseudo-SRS effect, the lineally decreasing SOD and nonlinear dispersion. The results were obtained by means of analytical method, based on evolution equations for the field moments, and verified by direct simulations. The stationary solitons exist due to the balance between the self-wavenumber downshift, caused by the pseudo-SRS, and the upshift induced by the decreasing SOD. The analytical solutions are close to their numerically found counterparts.

The present model does not take into regard the third-order linear dispersion. The compensation of the pseudo-SRS in the model of inhomogeneous media which includes this higher-order term will be considered elsewhere.

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References

[1] Infeld E and Rowlands G 2000 *Nonlinear Waves, Solitons and Chaos* (Cambridge: Cambridge University Press)
[2] Agrawal G P 2001 *Nonlinear Fiber Optic* (San Diego: Academic Press)
[3] Yang J 2001 *Solitons in Field Theory and Nonlinear Analysis* (New York: Springer)
[4] Kivshar Y S and Agraval G P 2003 *Optical Solitons: From Fibers to Photonic Crystals* (San Diego: Academic)
[5] Dickey L A 2005 *Soliton Equation and Hamiltonian Systems* (New York: World Scientific)
[6] Malomed B A 2006 *Soliton Management in Periodic Systems* (New York: Springer)
[7] Dauxois T and Peyrard M 2006 *Physics of Solitons* (Cambridge: Cambridge University Press)
[8] Zakharov V E and Shabat A B 1972 *Sov. Phys. JETP*. **34** 62-9

[9] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 142-4
[10] Gordon J P 1986 *Opt. Lett.* **11** 662-4
[11] Kodama Y 1985 *J. Stat. Phys.* **39** 597-614
[12] Kodama Y and Hasegawa A 1987 *IEEE J. Quant. Electron.* **23** 510-24
[13] Hong B and Lu D 2009 *Inter. J. Nonlin. Science* **7** 360-7
[14] Obregon M A and Stepanyants Yu A 1998 *Phys. Lett. A* **249** 315-23
[15] Kivshar Y S 1990 *Phys. Rev. A* **42** 1757-61
[16] Malomed B A and Tasgal R S 1998 *J. Opt. Soc. Am. B* **15** 162-70
[17] Biancalama F, Skrybin D V and Yulin A V 2004 *Phys. Rev. E* **70** 011615
[18] Essiambre R-J and Agrawal G P 1997 *J. Opt. Soc. Am. B* **14** 314-22
[19] Essiambre R-J and Agrawal G P 1997 *J. Opt. Soc. Am. B* **14** 323-30
[20] Andrianov A, Muraviev S, Kim A and Sysoliatin A 2007 *Laser Phys.* **17** 1296-302
[21] Chernikov S, Dianov E, Richardson D and Payne D 1993 *Opt. Lett.* **18** 476-8.
[22] Gromov E M and Malomed B A 2013 *J. Plasma Phys.* **79** 1057-62
[23] Gromov E M and Malomed B A 2014 *Opt. Comm.* **320** 88-93
[24] Aseeva N V, Gromov E M and Tyutin V V 2013 *Radiophys. Quant. Electr*. **56** 157-66